國立宜蘭大學 112 年度微積分競試 解答

1. $(-3,4)$	2. $(1,2)$
3. 0	4. 1
5. $\frac{2\sqrt{2}}{3}$	6. 1
7. $\frac{15}{2}$	8. $1-\frac{x^2}{2}+\frac{x^4}{4!}$
9. $-35\sin x - x\cos x$	10. $\frac{\sec^2 x-\cot x+1}{\csc x}$
11. $\frac{3^x\ln 3+4x^3}{\ln 2(3^x+x^4+5^6)}$	12. $\frac{\cot x}{\ln \sin x}$
13. $\frac{3}{2}\sqrt[3]{18}$	14. $\frac{3}{2}\ln\frac{2}{3}+\frac{\pi}{6}$
15. $f_x=4x^3+5y^3,\ f_y=15xy^2$	16. $f_x=y^2e^{xy},\ f_y=(1+xy)e^{xy}$
17. $\langle\cos y,\ -x\sin y\rangle$	18. $4e^2-4e+3$
19. $\frac{1}{2}\sin 1$	20. $\frac{2}{3}$
21. $2\left(\frac{1}{3}-\frac{1}{\pi}\right)$	22. $\frac{3}{4}$

1. $x+1$ 與 x^2+bx+c 均為連續的。只要使邊界點 $x=1$ 與 $x=3$ 左右端數值相同，保持連續性即可

$$f(1) = 1+b+c = 1+1 = 2 \qquad b+c = 1$$
$$f(3) = 9+3b+c = 3+1 = 4 \qquad 3b+c = -5 \qquad \text{解得 } b=-3,\ c=4$$

2.

The expression $\lim\limits_{x\to 0}\dfrac{a-\cos(bx)}{x^2}$ should be with indeterminate form $\frac{0}{0}$ to be with limit value 2.

$$a-\cos(b\times 0) = 0 \qquad \text{Applying L'Hopital's rule} \qquad \lim_{x\to 0}\frac{1-\cos(bx)}{x^2} = \lim_{x\to 0}\frac{b\sin(bx)}{2x}$$
$$a-1 = 0 \qquad = \lim_{x\to 0}\frac{b^2\cos(bx)}{2}$$
$$a = 1 \qquad = \frac{b^2}{2}$$

$\dfrac{b^2}{2} = 2$ since $b>0$. $b=2$. So $(a,b) = (1,2)$

3.

$$\lim_{x\to 1}\frac{(\ln x)^2}{x-1} = \lim_{x\to 1}\frac{2\ln x\cdot\frac{1}{x}}{1} \qquad \text{L'Hopital's Rule}$$
$$= \lim_{x\to 1}\frac{2\ln x}{x}$$
$$= \frac{2\ln 1}{1} = 0$$

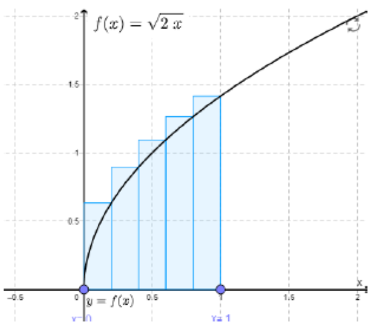
4.

$$y = \lim_{x\to\infty}(\ln x)^{\frac{2}{x}}$$
$$\ln y = \ln\left[\lim_{x\to\infty}(\ln x)^{\frac{2}{x}}\right] \qquad = \lim_{x\to\infty}\frac{2\ln(\ln x)}{x} \qquad \text{L'Hopital's rule}$$
$$= \lim_{x\to\infty}\ln\left[(\ln x)^{\frac{2}{x}}\right] \qquad \lim_{x\to\infty}\frac{2\cdot\frac{1}{\ln x}\frac{1}{x}}{1} \qquad \ln y = 0$$
$$= \lim_{x\to\infty}\frac{2}{x}\ln[(\ln x)] \qquad = \lim_{x\to\infty}\frac{2}{x\ln x} = 0 \qquad y = 1$$

5.

Consider the function $f(x)=\sqrt{2x}$. This problem indicates the **Riemann sum** of $f(x)$ over the range between $(0,1)$

$$\lim_{n\to\infty}\frac{1}{n}\left[\sqrt{\frac{2}{n}}+\sqrt{\frac{4}{n}}+\sqrt{\frac{6}{n}}+\cdots+\sqrt{\frac{2n}{n}}\right] = \int_0^1\sqrt{2x}dx$$
$$= \sqrt{2}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1$$
$$= \frac{2}{3}\sqrt{2}$$



6.

$$S_n = \sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\cdots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)}$$
$$= \left[\frac{1}{1}-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{4}\right]+\cdots\cdots+\left[\frac{1}{n-1}-\frac{1}{n}\right]+\left[\frac{1}{n}-\frac{1}{n+1}\right] = 1-\frac{1}{n+1}$$
$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \lim_{n\to\infty}S_n = 1$$

7.

$$\sum_{k=0}^{\infty}\frac{3\cdot 2^k+3^k}{5^k}=\sum_{k=0}^{\infty}\left(\frac{3\cdot 2^k}{5^k}+\frac{3^k}{5^k}\right)=3\sum_{k=0}^{\infty}\frac{2^k}{5^k}+\sum_{k=0}^{\infty}\frac{3^k}{5^k}=3\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k+\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k$$

$$=3\left[\frac{1}{1-\left(\frac{2}{5}\right)}\right]+\left[\frac{1}{1-\left(\frac{3}{5}\right)}\right]=3\left(\frac{5}{3}\right)+\frac{5}{2}=\frac{15}{2}$$

8.

$f(x)=cosx,\qquad f(0)=1$

$f'(x)=-\sin x,\qquad f'(0)=0$

$f''(x)=-\cos x,\qquad f'(0)=-1$

$f'''(x)=\sin x,\qquad f'''(0)=0$

$f^{(4)}(x)=\cos x,\qquad f^{(4)}(0)=1$

$$P_4(x)=1+0x+\frac{-1}{2}x^2+\frac{0}{3!}x^3+\frac{1}{4!}x^4=1-\frac{x^2}{2}+\frac{x^4}{4!}$$

9.

$$\frac{d}{dx}(x\cdot\sin x)=1\cdot\sin x+x\cdot\cos x$$

$$\frac{d^2}{dx^2}(x\cdot\sin x)=\frac{d}{dx}\left[\frac{d}{dx}(x\cdot\sin x)\right]=\frac{d}{dx}\left[\sin x+x\cdot\cos x\right]$$

$$=\cos x+\cos x+x\cdot(-\sin x)=2\cos x-x\sin x$$

$$\frac{d^3}{dx^3}(x\cdot\sin x)=\frac{d}{dx}\left[\frac{d^2}{dx^2}(x\sin x)\right]=\frac{d}{dx}\left[2\cos x-x\sin x\right]$$

$$=-2\sin x-(1\cdot\sin x+x\cos x)=-3\sin x-x\cos x$$

$$\frac{d^4}{dx^4}(x\cdot\sin x)=\frac{d}{dx}\left[-3\sin x-x\cdot\cos x\right]$$

$$=-3\cos x-(1\cdot\cos x+x\cdot(-\sin x))$$

$$=-4\cos x+x\sin x$$

$$\frac{d^5}{dx^5}(x\cdot\sin x)=5\sin x+x\cdot\cos x$$

由上式規則可推理，微分四次為一週期

$\because\ 35\div 4=8\cdots 3$

$\therefore\ \dfrac{d^{35}}{dx^{35}}(x\cdot\sin x)=-35\sin x-x\cos x$

10.

$$y'=\frac{\frac{d}{dx}(\tan x-1)\cdot\csc x-(\tan x-1)\cdot\frac{d}{dx}\csc x}{\csc^2 x}$$

$$=\frac{\sec^2x\cdot\csc x-(\tan x-1)\cdot(-1)\csc x\cdot\cot x}{\csc^2 x}$$

$$=\frac{\sec^2x+(\tan x-1)\cdot\cot x}{\csc x}=\frac{\sec^2x-\cot x+1}{\csc x}$$

11.

$$y = \log_2(3^x + x^4 + 5^6) \Rightarrow y = \frac{\ln(3^x + x^4 + 5^6)}{\ln 2}$$

$$y' = \frac{1}{\ln 2}\frac{d}{dx}\left[\ln(3^x + x^4 + 5^6)\right]$$

$$= \frac{1}{\ln 2} \cdot \frac{1}{3^x + x^4 + 5^6} \cdot \frac{d}{dx}(3^x + x^4 + 5^6)$$

$$= \frac{1}{\ln 2} \cdot \frac{1}{3^x + x^4 + 5^6} \cdot \left[\frac{d}{dx}3^x + 4x^3\right]$$

$$\frac{d}{dx}3^x = \frac{d}{dx}e^{\ln 3^x} = \frac{d}{dx}e^{x \cdot \ln 3} = e^{x \cdot \ln 3} \cdot \ln 3 = 3^x \cdot \ln 3$$

$$\therefore y' = \frac{1}{\ln 2(3^x + x^4 + 5^6)}(\ln 3 \cdot 3^x + 4x^3)$$

12.

$$y' = \frac{1}{\log_{10}\sin x} \cdot \frac{d}{dx}(\log_{10}\sin x)$$

$$= \frac{1}{\log_{10}\sin x} \cdot \frac{d}{dx}\left(\frac{\ln \sin x}{\ln 10}\right)$$

$$= \frac{1}{\log_{10}\sin x} \cdot \frac{1}{\ln 10} \cdot \left(\frac{1}{\sin x} \cdot \cos x\right)$$

$$= \frac{\ln 10}{\ln \sin x} \cdot \frac{1}{\ln 10} \cdot \cot x = \frac{\cot x}{\ln \sin x}$$

13.

$u = x^2 - 4, du = 2xdx \quad \int_2^4 \frac{x}{\sqrt[3]{x^2 - 4}}dx = \int_0^{12} \frac{1}{2}u^{\frac{-1}{3}}du = \frac{3}{4}u^{\frac{2}{3}}\Big|_0^{12} = \frac{3}{2}\sqrt[3]{18}$

14.

Let

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}$$

From the method of coefficient comparison,

we have $a = -3$, $b = 3$, $c = 2$

Then integral becomes

$$\int_1^{\sqrt{3}} \frac{-3dx}{x} + \int_1^{\sqrt{3}} \frac{3x dx}{x^2+1} + \int_1^{\sqrt{3}} \frac{2dx}{x^2+1}$$

$$= \left[-3\ln|x| + \frac{3}{2}\ln(x^2+1) + 2\arctan(x)\right]\Big|_1^{\sqrt{3}}$$

$$= \frac{3}{2}\ln\left(\frac{x^2+1}{x^2}\right)\Big|_1^{\sqrt{3}} + 2(\arctan\sqrt{3} - \arctan 1)$$

$$= \frac{3}{2}\left(\ln\frac{4}{3} - \ln 2\right) + 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{3}{2}\ln\frac{2}{3} + \frac{\pi}{6}$$

15.

$$f(x,y)=x^4+5xy^3 \quad \Rightarrow \quad f_x(x,y)=\underline{4x^3+5y^3},\ f_y(x,y)=0+5x\cdot 3y^2=\underline{15xy^2}$$

16.

$$f(x,y)=ye^{xy} \quad \Rightarrow \quad f_x(x,y)=y\cdot e^{xy}\cdot y=\underline{y^2e^{xy}},\ f_y(x,y)=y\cdot e^{xy}\cdot x+e^{xy}\cdot 1=\underline{e^{xy}+xye^{xy}}$$

17.

$$\nabla f(x,y)=\left[\frac{\partial}{\partial x}(x\cos y)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(x\cos y)\right]\mathbf{j}=\cos y\,\mathbf{i}-x\sin y\,\mathbf{j}$$

18.

$$\int_1^2\int_0^2(y+2xe^y)\,dx\,dy=\int_1^2\left[xy+x^2e^y\right]_{x=0}^{x=2}dy=\int_1^2(2y+4e^y)\,dy=\left[y^2+4e^y\right]_1^2$$

$$=4+4e^2-1-4e=\underline{4e^2-4e+3}$$

19.

$$\int_0^1\int_0^x\cos(x^2)\,dy\,dx=\int_0^1\left[\cos(x^2)y\right]_{y=0}^{y=x}dx=\int_0^1 x\cos(x^2)\,dx=\tfrac{1}{2}\sin(x^2)\Big]_0^1=\underline{\tfrac{1}{2}\sin 1}$$

20.

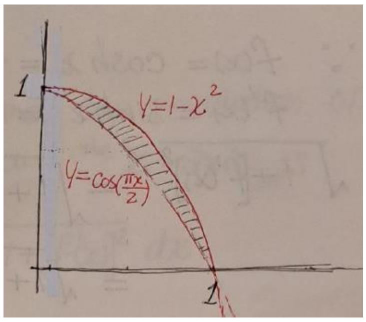
$$\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}}y\sin x\,dz\,dy\,dx=\int_0^\pi\int_0^1\left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}}dy\,dx=\int_0^\pi\int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$$

$$=\int_0^\pi\left[-\tfrac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1}dx=\int_0^\pi\tfrac{1}{3}\sin x\,dx=-\tfrac{1}{3}\cos x\Big]_0^\pi=\underline{\tfrac{2}{3}}$$

21.

$$A=\int_0^1\left[1-x^2-\cos\left(\frac{\pi x}{2}\right)\right]dx$$

$$=\left[x-\frac{x^3}{3}-\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]\Bigg|_0^1=2\left(\frac{1}{3}-\frac{1}{\pi}\right)$$



22.

$$f'(x)=\frac{e^x-e^{-x}}{2}$$

$$1+[f'(x)]^2=1+\frac{e^{2x}-2+e^{-2x}}{4}=\frac{e^{2x}+2+e^{-2x}}{4}=\left(\frac{e^x+e^{-x}}{2}\right)^2$$

$$s=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

$$s=\int_0^{\ln 2}\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}\,dx=\int_0^{\ln 2}\frac{e^x+e^{-x}}{2}dx=\frac{e^x-e^{-x}}{2}\Bigg|_0^{\ln 2}$$

$$=\frac{1}{2}\left(2-\frac{1}{2}\right)-\frac{1}{2}(1-1)=\frac{3}{4}$$