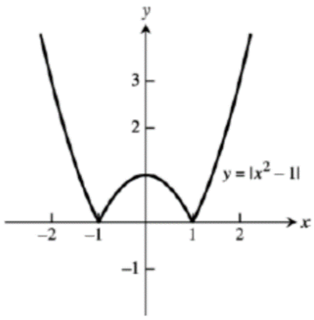
國立宜蘭大學 112 年度第二次微積分競試 解答

1. $\sqrt{y^4 - y^2 + 1}$	2. 
3. 7	4. 7
5. diverges	6. diverges
7. $x - 2y - 2z = -5$	8. $x = -t + 1, y = 2t + 1, z = 2t + 2$
9. $\frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x}$	10. $e^{-\frac{x^2}{2}}(x^2 - 1)$
11. $x^{x-2}(x\ln x + x - 1)$	12. $\frac{\pi}{3} - \sqrt{3}$
13. 2	14. $\frac{2}{9}\sqrt{(x^3 + 1)^3} + C$
15. $-\ln 2 - \frac{1}{3}\ln 5 - \frac{1}{2}$	16. $\frac{1}{6}(e^9 - 1)$
17. $\langle 6, 9 \rangle$	18. $\frac{44}{3}$
19. $\frac{13\pi}{3}$	20. $x = 6t + 1, y = t + 2, z = 10t + 1$
21. $(-\sqrt{3},\ 1,\ 2 + \sqrt{3})$	22. $\frac{1}{2}e^{16} + \frac{1}{4}e^8 + 3e^2 + \frac{57}{4}$

1.

$$y=\sqrt{x-3}\Rightarrow y^2+3=x; L=\sqrt{(x-4)^2+(y-0)^2}=\sqrt{(y^2+3-4)^2+y^2}=\sqrt{(y^2-1)^2+y^2}$$
$$=\sqrt{y^4-2y^2+1+y^2}=\sqrt{y^4-y^2+1}$$

2.

$y=x^2-1$ 為頂點(0, -1)開口向上二次曲線，取絕對值後，x 軸下方圖形(-1 < x < 1)翻轉到上方。

3.

$$\lim_{x\to\infty}\frac{7x^3}{x^3-3x^2+6x}=\lim_{x\to\infty}\frac{7}{1-\frac{3}{x}+\frac{9}{x^2}}=7$$

4.

$$1=\lim_{x\to4}\frac{f(x)-5}{x-2}=\frac{\lim_{x\to4}f(x)-\lim_{x\to4}5}{\lim_{x\to4}x-\lim_{x\to4}2}=\frac{\lim_{x\to4}f(x)-5}{4-2}\Rightarrow\lim_{x\to4}f(x)-5=2(1)\Rightarrow\lim_{x\to4}f(x)=2+5=7.$$

5.

Use the Limit Comparison Test with $a_n=\dfrac{n^2-1}{n^3+1}$ and $b_n=\dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{(n^2-1)n}{n^3+1}=\lim_{n\to\infty}\frac{n^3-n}{n^3+1}=\lim_{n\to\infty}\frac{1-1/n^2}{1+1/n^3}=1>0$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series, the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-1}{n^3+1}$ also diverges.

6.

$\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{2x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{2}=\infty$, so $\displaystyle\lim_{n\to\infty}\frac{e^n}{n^2}=\infty$. Thus, the series $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^2}$ diverges by the Test for Divergence.

7.

The equation of the surface:

$f(x,y,z)=x^2+z^2e^{y-x}=5$

$\nabla f=(2x-z^2e^{y-x})\hat{i}+(z^2e^{y-x})\hat{j}+(2ze^{y-x})\hat{k}$

$\nabla f(1,1,2)=(2\cdot1-2^2\cdot e^{1-1})\hat{i}+(2^2e^{1-1})\hat{j}+(2\cdot2\cdot e^{1-1})\hat{k}$

$=(-2)\hat{i}+4\hat{j}+4\hat{k}$

$=2[(-\hat{i})+2\hat{j}+2\hat{k}]$

∴ (2) The equation of the tangent plane is

$-(x-1)+2(y-1)+2(z-2)=0$

$x-2y-2z=-5$

8.

$x=-t+1,\qquad y=2t+1,\qquad z=2t+2$

9.

$$y' = \frac{\frac{d}{dx}[x^3](\cos x) - \frac{d}{dx}[\cos x] \cdot x^3}{\cos^2 x} = \frac{3x^2 \cdot \cos x - (-\sin x) \cdot x^3}{\cos^2 x} = \frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x}$$

10.

$$y' = e^{-\frac{x^2}{2}}(-x) = -xe^{-\frac{x^2}{2}}$$

$$y'' = -e^{-\frac{x^2}{2}} - x\frac{d}{dx}\left[e^{-\frac{x^2}{2}}\right] = \left(-1 + x^2\right) e^{-\frac{x^2}{2}} = \left(x^2 - 1\right) e^{-\frac{x^2}{2}}$$

11. 取對數再微分

$$\ln y = (x-1)\ln x \qquad \frac{1}{y}y' = \ln x + (x-1) \times \frac{1}{x}$$

$$y' = y\left(\ln x + \frac{x-1}{x}\right) = x^{x-1}\left(\ln x + \frac{x-1}{x}\right) = x^{x-2}\left(x \ln x + x - 1\right)$$

12.

$$f(x) = x - 2\sin x \qquad f'(c) = 1 - 2\cos c = 0 \quad \Rightarrow \quad \cos c = \frac{1}{2}$$

$$f'(x) = 1 - 2\cos x \qquad \Rightarrow \quad c = \frac{\pi}{3}$$

$c = 0$	$c = \frac{\pi}{3}$	$c = \pi$
$f(0) = 0$	$f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$	$f(2\pi) = 2\pi$

The minimum value is $\frac{\pi}{3} - \sqrt{3}$

13.

$$(4-x)y^2 = x^3 \qquad \frac{d}{dx}(4-x) \cdot y^2 + (4-x) \cdot 2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(4-x)} \qquad \frac{dy}{dx} = \frac{3 \times 2^2}{2 \times 2 \times (4-2)} = 2$$

14.

Let $u = x^3 + 1$. Then $du = 3x^2\, dx$ and $x^2\, dx = \frac{1}{3}\, du$, so

$$\int x^2 \sqrt{x^3+1}\, dx = \int \sqrt{u}\left(\tfrac{1}{3}\, du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3+1)^{3/2} + C.$$

15.

$\frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$. Multiplying both sides by $(3x-1)(x-1)^2$ gives

$x(3-5x) = A(x-1)^2 + B(x-1)(3x-1) + C(3x-1)$. Substituting 1 for x gives $-2 = 2C \Leftrightarrow C = -1$.

Substituting $\frac{1}{3}$ for x gives $\frac{4}{9} = \frac{4}{9}A \Leftrightarrow A = 1$. Substituting 0 for x gives $0 = A + B - C = 1 + B + 1$, so $B = -2$.

Thus,

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\, dx = \int_2^3 \left[\frac{1}{3x-1} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\right] dx = \left[\frac{1}{3}\ln|3x-1| - 2\ln|x-1| + \frac{1}{x-1}\right]_2^3$$

$$= \left(\tfrac{1}{3}\ln 8 - 2\ln 2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{3}\ln 5 - 0 + 1\right) = -\ln 2 - \tfrac{1}{3}\ln 5 - \tfrac{1}{2}$$

16.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx dy = \int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\, dy dx$$

$$= \int_0^3 y e^{x^2}\Big|_0^{\frac{1}{3}x} dx == \int_0^3 \frac{1}{3}xe^{x^2}\, dx = \frac{1}{6}e^{x^2}\Big|_0^3 = \frac{1}{6}(e^9 - 1)$$

17.

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xe^y\,\mathbf{i} + x^2e^y\,\mathbf{j}$$

$$\nabla f(3,0) = 2(3)e^0\,\mathbf{i} + 3^2e^0\,\mathbf{j} = 6\,\mathbf{i} + 9\,\mathbf{j}$$

18.

$$A = \int_{-2}^{1}\left[\left(\frac{2}{3}x + \frac{16}{3}\right) - x^2\right]dx + \int_{1}^{2}\left[(-2x+8) - x^2\right]dx$$

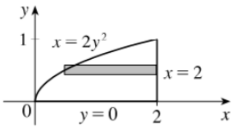
$$= \left[\frac{1}{3}x^2 + \frac{16}{3}x - \frac{1}{3}x^3\right]_{-2}^{1} + \left[-x^2 + 8x - \frac{1}{3}x^3\right]_{1}^{2}$$

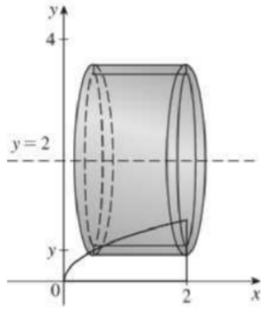
$$= \left[\left(\frac{1}{3} + \frac{16}{3} - \frac{1}{3}\right) - \left(\frac{4}{3} - \frac{32}{3} + \frac{8}{3}\right)\right] + \left[\left(-4 + 16 - \frac{8}{3}\right) - \left(-1 + 8 - \frac{1}{3}\right)\right] = \frac{44}{3}$$

19.

The shell has radius $2-y$, circumference $2\pi(2-y)$, and height $2-2y^2$.

$$V = \int_0^1 2\pi(2-y)(2-2y^2)\,dy$$
$$= 4\pi\int_0^1 (2-y)(1-y^2)\,dy$$
$$= 4\pi\int_0^1 (y^3 - 2y^2 - y + 2)\,dy$$
$$= 4\pi\left[\tfrac{1}{4}y^4 - \tfrac{2}{3}y^3 - \tfrac{1}{2}y^2 + 2y\right]_0^1$$
$$= 4\pi\left(\tfrac{1}{4} - \tfrac{2}{3} - \tfrac{1}{2} + 2\right)$$
$$= 4\pi\left(\tfrac{13}{12}\right) = \tfrac{13\pi}{3}$$





20.

we need to determine the vector $\vec{V_T}$ of the tangent line.

Since $\vec{V_T} \perp \nabla f$, and, $\vec{V_T} \perp \nabla g$, it can be concluded that $\vec{V_T} \parallel (\nabla f \times \nabla g)$.

$$\nabla f(x,y,z) = 2(3x\,\hat{i} + y\,\hat{j} + 2z\,\hat{k})$$
$$\nabla g(x,y,z) = 2(x\,\hat{i} + 2y\,\hat{j} + z\,\hat{k})$$
$$\nabla f(1,2,1) \times \nabla g(1,2,1)$$
$$= 4\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 2 \\ 1 & 4 & 1 \end{vmatrix}$$
$$= 4\left[(2\hat{i} + 2\hat{j} + 12\hat{k}) - (8\hat{i} + 3\hat{j} + 2\hat{k})\right]$$
$$= 4(-6\hat{i} - \hat{j} + 10\hat{k})$$
$$\therefore\ \vec{V_T} = k'(-6\hat{i} - \hat{j} + 10\hat{k}) = k(6\hat{i} + \hat{j} - 10\hat{k})$$

$$\begin{cases} x(t) = 6t + 1 \\ y(t) = t + 2 \\ z(t) = -10t + 1 \end{cases}$$

21.

$f(x,y,z) = x + y + z - 3 = 0$

$g(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2 = 9$

$\Big\}$ curve of the intersection of $f(x,y,z)=0$ and $g(x,y,z)=9$

$h(x,y,z) = x^2 + y^2 + z^2$

(objective function for evaluating the distance to the origin, since the distance from (x,y,z) to the origin is $\sqrt{h(x,y,z)}$.)

There may be two constants λ and μ so that $P(a,b,c)$ satisfies the following equation

$$\lambda \nabla f(a,b,c) + \mu \nabla g(a,b,c) = \nabla h(a,b,c)$$

However

$$\nabla f(x,y,z) = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla g(x,y,z) = 2\left[(x-1)\hat{i} + (y-2)\hat{j} + (z-3)\hat{k}\right]$$

$$\nabla h(x,y,z) = 2\left[(x)\hat{i} + (y)\hat{j} + (z)\hat{k}\right]$$

$\Rightarrow$

$$\begin{cases} \lambda + 2\mu(a-1) = 2a \\ \lambda + 2\mu(b-2) = 2b \\ \lambda + 2\mu(c-3) = 2c \end{cases}$$

Solving a, b, and c in terms of μ and λ yields

$$\left.\begin{aligned} a &= \frac{1}{2}\left(\frac{-\lambda+2\mu}{\mu-1}\right) \\ b &= \frac{1}{2}\left(\frac{-\lambda+4\mu}{\mu-1}\right) \\ c &= \frac{1}{2}\left(\frac{-\lambda+6\mu}{\mu-1}\right) \end{aligned}\right\} \quad \text{—— (1)}$$

Recall that $P(a,b,c)$ on the curve of intersection

$$\begin{cases} a+b+c = 3 & \text{—— (2)} \\ (a-1)^2+(b-2)^2+(c-3)^2 = 9 & \text{—— (3)} \end{cases}$$

Substituting (1) into (2) yields

$$\frac{3}{2}\ \frac{(-\lambda+4\mu)}{\mu-1} = 3$$

$\Rightarrow$ $-\lambda + 4\mu = 2(\mu-1)$

or $\lambda = 2(\mu+1)$

$\therefore$

$a = \frac{-1}{\mu-1}$

$b = 1$

$c = \frac{2\mu-1}{\mu-1}$

$\Rightarrow$ $a+c=2$

or $c = 2-a$

$$\left.\begin{array}{l} a \\ b=1 \\ c=2-a \end{array}\right\} \text{Substitution into } (a-1)^2+(b-2)^2+(c-3)^2=9$$

$\Rightarrow$ $(a-1)^2+(1-2)^2+[(2-a)-3]^2=9$

$a^2-2a+1+1+a^2+2a+1=9$

$2a^2=6$

$a=\sqrt{3}$ or $-\sqrt{3}$

$c=2-\sqrt{3}$ or $2+\sqrt{3}$

$$\begin{cases} a=\sqrt{3} \\ b=1 \\ c=2-\sqrt{3} \end{cases} \text{ or } \begin{cases} a=-\sqrt{3} \\ b=1 \\ c=2+\sqrt{3}. \end{cases}$$

$\Rightarrow$ $a^2+b^2+c^2=11-4\sqrt{3}$ or $a^2+b^2+c^2=11+4\sqrt{3}$

$\Rightarrow$ The farthest point $P(a,b,c)=(-\sqrt{3},\ 1,\ 2+\sqrt{3})$

22.

Since the path of $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

$= t^3\,\hat{i} + (1-3t)\,\hat{j} + e^t\,\hat{k} \quad (0 \le t \le 2)$

then we have

$$d\vec{r}(t) = \vec{r}\,'(t)\,dt$$

$$= \left[(3t^2)\,\hat{i} + (-3)\,\hat{j} + e^t\,\hat{k}\right] dt$$

$$\vec{F}(x, y, z) = \vec{F}(x(t), y(t), z(t))$$

$$= e^{2(t^3)}\,\hat{i} + e^t\left[(1-3t)+1\right]\hat{j} + (e^t)^3\,\hat{k}$$

$$= e^{2t^3}\,\hat{i} + (2-3t)\,e^t\,\hat{j} + e^{3t}\,\hat{k}$$

$$\vec{F} \cdot \vec{r}\,'(t) = 3t^2 \cdot e^{2t^3} - 3(2-3t)e^t + e^{4t}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 3t^2 e^{2t^3}\,dt - 6\int_0^2 e^t\,dt + 9\int_0^2 t e^t\,dt + \int_0^2 e^{4t}\,dt$$

$$= \frac{1}{2}\int_0^2 e^{2t^3} \cdot d(2t^3) - 6\int_0^2 d(e^t) + 9\int_0^2 t\,d(e^t) + \frac{1}{4}\int_0^2 e^{4t} d(4t)$$

$$= \frac{1}{2}\left(e^{2t^3}\Big|_0^2\right) - 6\left(e^t\Big|_0^2\right) + 9\left[(te^t - e^t)\Big|_0^2\right] + \frac{1}{4}\left(e^{4t}\Big|_0^2\right)$$

$$= \frac{1}{2}(e^{16}-1) - 6(e^2-1) + 9\left[(2e^2-e^2)-(0\cdot1-1)\right] + \frac{1}{4}(e^8-1)$$

$$= \frac{1}{2}e^{16} + \frac{1}{4}e^8 + 3e^2 + \frac{57}{4}$$