國立宜蘭大學 100 學年度微積分競試解答

1. Evaluate $\cos(\tan^{-1}(\sin(\cot^{-1}x)))=?$

(A) $\sqrt{\frac{x^2+1}{x^2+2}}$ (B) $\sqrt{\frac{x^2+3}{x^2+2}}$ (C) $\sqrt{\frac{x^2+3}{x^2+1}}$ (D) $\frac{x^2+1}{x^2+2}$ (E) $\sqrt[3]{\frac{x^2+1}{x^2+2}}$

Sol: $Let:\theta=\cot^{-1}x, \cot\theta=x$

$$\sin\theta=\frac{1}{\sqrt{1+x^2}}$$

$$Let:\alpha=\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right), \tan\alpha=\left(\frac{1}{\sqrt{1+x^2}}\right)\Rightarrow\cos\alpha=\sqrt{\frac{1+x^2}{2+x^2}}$$

Ans:(A)

2. Evaluate $\lim_{x\to\infty}\frac{3^x+4^x}{3^{x+1}+4^{x+2}}=?$

(A) $\frac{1}{8}$ (B) $\frac{1}{16}$ (C) $\frac{1}{24}$ (D) $\frac{1}{32}$ (E) $\frac{1}{64}$

Sol: $\lim_{x\to\infty}\frac{3^x+4^x}{3^{x+1}+4^{x+2}}=\lim_{x\to\infty}\frac{\left(\frac{3}{4}\right)^x+1}{3\left(\frac{3}{4}\right)^x+16}=\frac{1}{16}$

Ans:(B)

3. Evaluate $\lim_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)=?$

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Sol: $\lim_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)=\lim_{x\to\infty}\frac{\left(e^{\frac{1}{x}}-1\right)}{\frac{1}{x}}$ $\quad Let:y=\frac{1}{x}\Rightarrow\left(x\to\infty\Leftrightarrow y\to0^+\right)$

$$=\lim_{y\to0^+}\frac{\left(e^y-1\right)}{y}=\lim_{y\to0^+}\frac{e^y}{1}=1$$

Ans:(D)

4. Find $\lim_{\theta\to0}\frac{\sin\theta}{\theta+\tan\theta}=?$

(A) $\frac{3}{4}$ (B) $\frac{1}{4}$ (C) $\frac{2}{5}$ (D) $\frac{2}{3}$ (E) $\frac{1}{2}$

Sol: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\frac{\sin\theta}{\cos\theta}}=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\theta}}{1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}}=\frac{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$

Ans:(E)

5. Evaluate $\frac{d}{dx}(9^x)=?$

(A) $x\cdot 9^{x-1}$ (B) $9^x\cdot\ln 9$ (C) $9^x\cdot\log 9$ (D) 9^x (E) $x\cdot 9^x$

Sol: $\frac{d}{dx}(9^x)=9^x\cdot\ln 9$

Ans:(B)

6.Assume $f(x)=(\sin 10x+\cos 10x)^{10}$, please find $f^{/}(\pi)=?$

(A) 100 (B) 110 (C) 0 (D) 10 (E) 111

Sol: $f^{/}(x)=10(\sin 10x+\cos 10x)^9\times(10\cos 10x-10\sin 10x)\Rightarrow f^{/}(\pi)=100$

Ans:(A)

7. Assume $f(x)=\frac{1}{x^2-1}$, please find $f^{(100)}(0)=?$

(A) 0 (B) 100! (C) $-100!$ (D) 100 (E) -100

Sol: $f(x)=\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$\Rightarrow f^{/}(x)=\frac{1}{2}\left[-(x-1)^{-2}-(-1)(x+1)^{-2}\right]$

$\Rightarrow f^{//}(x)=\frac{1}{2}\left[(-1)(-2)(x-1)^{-3}-(-1)(-2)(x+1)^{-3}\right]$

$\Rightarrow f^{///}(x)=\frac{1}{2}\left[(-1)^3 3!(x-1)^{-4}-(-1)^3 3!(x+1)^{-4}\right]$

$\vdots$

$\Rightarrow f^{(n)}(x)=\frac{1}{2}\left[(-1)^n n!(x-1)^{-(n+1)}-(-1)^n n!(x+1)^{-(n+1)}\right]=\frac{1}{2}(-1)^n n!\left[\frac{1}{(x-1)^{n+1}}-\frac{1}{(x+1)^{n+1}}\right]$

$\therefore f^{(100)}(0)=\frac{1}{2}(-1)^{100}\times 100!\left[\frac{1}{(-1)^{101}}-\frac{1}{1^{101}}\right]=-100!$

Ans:(C)

8.Assume $f(x)=\sqrt{\frac{(x-1)(x-2)^2(x-3)^3}{x^2+4}}$, please find $f^{/}(0)=?$

(A) $-\frac{9\sqrt{3}}{2}$ (B) $-\frac{9\sqrt{2}}{2}$ (C) $\frac{9\sqrt{2}}{2}$ (D) $\frac{9\sqrt{3}}{2}$ (E) $5\sqrt{3}$

Sol: $\ln f(x)=\ln\left[\dfrac{(x-1)(x-2)^2(x-3)^3}{x^2+4}\right]^{\frac{1}{2}}=\dfrac{1}{2}\left[\ln(x-1)+2\ln(x-2)+3\ln(x-3)-\ln(x^2+4)\right]$

$$\Rightarrow \frac{d\ln f(x)}{dx}=\frac{1}{f(x)}\cdot f'(x)=\frac{1}{2}\left[\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}-\frac{2x}{x^2+4}\right]$$

$$\Rightarrow f'(x)=f(x)\frac{1}{2}\left[\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}-\frac{2x}{x^2+4}\right]$$

$$\therefore f'(0)=\frac{f(0)}{2}\left[-1-1-1-0\right]=-\frac{3}{2}\cdot f(0)=-\frac{3}{2}\cdot 3\sqrt{3}=-\frac{9\sqrt{3}}{2}$$

Ans:(A)

9. Evaluate $\int_{-2}^{2}|x+1|dx=?$

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Sol: $\int_{-2}^{2}|x+1|dx=\int_{-2}^{-1}(-x-1)dx+\int_{-1}^{2}(x+1)dx=\left[\dfrac{-x^2}{2}-x\right]_{-2}^{-1}+\left[\dfrac{x^2}{2}+x\right]_{-1}^{2}=\left(\dfrac{1}{2}-0\right)+\left(4+\dfrac{1}{2}\right)=5$

Ans:(C)

10. Evaluate $\int_0^{\pi}x^2\sin xdx=?$

(A) $\pi+4$ (B) π^2+4 (C) 0 (D) π^2-4 (E) $\pi-4$

Sol: $\int_0^{\pi}x^2\sin xdx$

$$=-\int_0^{\pi}x^2d\cos x$$

$$=-x^2\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos xdx^2$$

$$=\pi^2+\int_0^{\pi}\cos x(2xdx)$$

$$=\pi^2+2\int_0^{\pi}x\cos xdx$$

$$=\pi^2+2\int_0^{\pi}xd\sin x$$

$$=\pi^2+2\left[x\sin x\Big|_0^{\pi}-\int_0^{\pi}\sin xdx\right]$$

$$=\pi^2+2\left[0-\cos x\Big|_0^{\pi}\right]$$

$$=\pi^2+2\times(-2)$$

$$=\pi^2-4$$

Ans:(D)

11. Evaluate $\int_0^{\pi}\sin 3x\cos 5xdx=?$

(A) π (B) $2\pi^2+4$ (C) 0 (D) $-\pi$ (E) $2\pi^2-4$

Sol: $\int_0^{\pi} \sin 3x \cos 5x dx$

$$= \int_0^{\pi} \cos 5x \sin 3x dx$$
$$= \frac{1}{2}\int_0^{\pi} (\sin 8x - \sin 2x) dx$$
$$= \frac{1}{2}\left(-\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x\right)\Bigg|_0^{\pi}$$
$$= 0$$

Ans:(C)

12. Evaluate $\int_{-1}^{3} [x] dx = ?$

(A) 0 (B) 2 (C) 4 (D) 6 (E) 7

Sol: $\int_{-1}^{3} [x] dx$

$$= \int_{-1}^{0} [x] dx + \int_{0}^{1} [x] dx + \int_{1}^{2} [x] dx + \int_{2}^{3} [x] dx$$
$$= \int_{-1}^{0} (-1) dx + \int_{0}^{1} 0 dx + \int_{1}^{2} 1 dx + \int_{2}^{3} 2 dx$$
$$= -1 + 0 + 1 + 2 = 2$$

Ans:(B)

13. Evaluate $\int_{10}^{100} \frac{1}{x(\log x)^2} dx = ?$

(A) $\frac{1}{2}\log 100$ (B) $\frac{1}{2}\log 10$ (C) $\frac{1}{4}\log 10$ (D) $\frac{1}{2}\ln 100$ (E) $\frac{1}{2}\ln 10$

Sol: $Let : u = \log x \Rightarrow du = \frac{1}{\ln 10} \cdot \frac{1}{x} dx$

$$\therefore \int_{10}^{100} \frac{1}{x(\log x)^2} dx = \int_{1}^{2} \frac{1}{u^2} (\ln 10) du = (\ln 10)\left(-\frac{1}{u}\right)\Bigg|_1^2 = \frac{1}{2}\ln 10$$

Ans:(E)

14. Evaluate $\int_{-5}^{5} \frac{\sin x}{\sqrt{1+x^4}} dx = ?$

(A) $-5\sqrt{5}$ (B) 0 (C) $5\sqrt{5}$ (D) $5+\sqrt{5}$ (E) $5-\sqrt{5}$

Sol: $Let : f(x) = \frac{\sin x}{\sqrt{1+x^4}}$

$$\because f(-x) = \frac{\sin(-x)}{\sqrt{1+(-x)^4}} = -\frac{\sin x}{\sqrt{1+x^4}} = -f(x)$$
$$\therefore f(x) \Rightarrow \text{odd function} \Rightarrow \int_{-5}^{5} \frac{\sin x}{\sqrt{1+x^4}} dx = 0$$

Ans:(B)

15. Evaluate $\int \frac{1}{1-\sin x}dx = ?$

(A) $\sin x + \sec x + C$ (B) $\cos x + \sec x + C$ (C) $\tan x + \sec x + C$ (D) $\tan x + \cot x + C$ (E) 0

Sol: $\int \frac{1}{1-\sin x}dx = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}dx = \int \frac{1+\sin x}{\cos^2 x}dx = \int (\sec^2 x + \sec x \tan x)dx = \tan x + \sec x + C$

Ans:(C)

16. Evaluate $\int \frac{x+1}{x^2-3x+2}dx = ?$

(A) $\ln|x^2-3x+2|+C$ (B) $-\ln|x-1|+2\ln|x-2|+C$ (C) $-2\ln|x-1|+3\ln|x-2|+C$

(D) $\ln|x^2+3x-2|+C$ (E) $\ln\left|\frac{x+1}{x^2-3x+2}\right|+C$

Sol: $Let: \frac{x+1}{x^2-3x+2} = \frac{x+1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} \Leftrightarrow x+1 = A(x-2)+B(x-1)$

$$\begin{cases} x=1 \Rightarrow A=-2 \\ x=2 \Rightarrow B=3 \end{cases}$$

$$\therefore \int \frac{x+1}{x^2-3x+2}dx = \int \left(\frac{-2}{(x-1)} + \frac{3}{(x-2)} \right) dx = -2\ln|x-1| + 3\ln|x-2| + C$$

Ans:(C)

17. $x^2 - xy + y^2 = 1$, Evaluate $\left.\frac{dy}{dx}\right|_{(1,1)} = ?$

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Sol: $Let: f(x,y) = x^2 - xy + y^2 - 1 = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2x-y}{-x+2y}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(1,1)} = -1$$

Ans:(B)

18. Find the area of the region bounded by $x = 3 - y^2$ and $x = y + 1$.

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{9}{2}$ (E) 2

Sol: $Let: 3 - y^2 = y + 1$

$\Rightarrow y^2 + y - 2 = 0$

$\Rightarrow (y-1)(y+2) = 0$

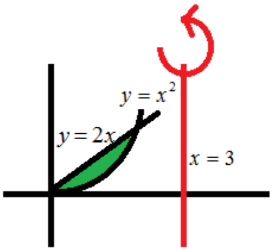
$\Rightarrow y = 1, -2$ Ans:(D)

$\Rightarrow x = 2, -1$

$$\therefore Area = A = \int_{-1}^{2}\left[(x-1)-(-\sqrt{3-x})\right]dx + \int_{2}^{3}\sqrt{3-x}\,dx = \left(\frac{x^2}{2} - x - \frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right)_{-1}^{2} + 2\times\left(-\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right)_{2}^{3} = \frac{9}{2}$$

19.The region enclosed by $y = x^2$ and $y = 2x$ is rotated about the line $x = 3$. Find the volume of the resulting solid.

(A) $\frac{16}{3}\pi$ (B) $\frac{16}{3}$ (C) $\frac{16}{7}\pi$ (D) $\frac{16}{7}$ (E) π



Sol: $Let : x^2 = 2x \Rightarrow x = 0,2$

$\therefore point(0,0) \,\&\, (2,4)$

$V = \int_0^2 2\pi r h dx$

$= \int_0^2 2\pi(3-x)(2x - x^2)dx$

$= 2\pi\int_0^2 (6x - 5x^2 + x^3)dx = \frac{16}{3}\pi$

Ans:(A)

20. Evaluate $\int e^{ax}\sin bx dx = ?$

(A) $\frac{e^{ax}}{a^2+b^3}(a\sin bx - b\cos bx) + C$ (B) $\frac{e^{bx}}{a^2+b^2}(a\sin ax - b\cos bx) + C$

(C) $\frac{e^{ax}}{a^2+b^2}(a\sin bx + b\cos bx) + C$ (D) $\frac{e^{bx}}{a^2+b^2}(a\sin bx - a\cos bx) + C$

(E) $\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

Sol: $\int e^{ax}\sin bx dx$

$$= \frac{1}{a}\int \sin bx de^{ax}$$

$$= \frac{1}{a}\left[e^{ax}\sin bx - \int e^{ax} d\sin bx\right]$$

$$= \frac{1}{a}\left[e^{ax}\sin bx - b\int e^{ax}\cos bx dx\right]$$

$$= \frac{1}{a}\left[e^{ax}\sin bx - \frac{b}{a}\int \cos bx de^{ax}\right]$$

$$= \frac{1}{a}\left[e^{ax}\sin bx - \frac{b}{a}\left(e^{ax}\cos bx - \int e^{ax} d\cos bx\right)\right]$$

$$= \frac{ae^{ax}\sin bx}{a^2} - \frac{b}{a^2}\left(e^{ax}\cos bx - \int e^{ax} d\cos bx\right)$$

$$= \frac{ae^{ax}\sin bx}{a^2} - \frac{b}{a^2}\left(e^{ax}\cos bx + b\int e^{ax}\sin bx dx\right)$$

$$= \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\sin bx dx$$

$$\Rightarrow \int e^{ax}\sin bx dx = \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\sin bx dx$$

$$\Rightarrow (a^2+b^2)\int e^{ax}\sin bx dx = ae^{ax}\sin bx - be^{ax}\cos bx$$

$$\Rightarrow \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Ans:(E)

21.Evaluate $\frac{d}{dx}\int_1^{\sin x}\frac{1}{1+t^4}dt = ?$

(A) $\frac{\cos x}{1+\cos^4 x}$ (B) $\frac{\cos x}{1-\sin^4 x}$ (C) $\frac{\sin x}{1+\cos^4 x}$ (D) $\frac{\sin x}{1+\sin^4 x}$ (E) $\frac{\cos x}{1+\sin^4 x}$

Sol: $\frac{d}{dx}\int_{p(x)}^{q(x)} f(t)dt = f(q(x))\cdot q'(x) - f(p(x))\cdot p'(x)$

$$\Rightarrow \frac{d}{dx}\int_1^{\sin x}\frac{1}{1+t^4}dt = \frac{1}{1+\sin^4 x}\cdot\cos x - 0 = \frac{\cos x}{1+\sin^4 x}$$ Ans:(E)

22.Evaluate $\int \sin^{-1} x dx = ?$

(A) $x^2\sin^{-1}x + \sqrt{1+x^2} + C$ (B) $x\cos^{-1}x + \sqrt{1-x^2} + C$ (C) $x\sin^{-1}x + \sqrt{1+x^2} + C$

(D) $x\sin^{-1}x + \sqrt{1-x^2} + C$ (E) $x\sin x + \sqrt{1+x^2} + C$

Sol: $\int \sin^{-1} x dx = x\sin^{-1}x - \int x d\sin^{-1}x$

$$= x\sin^{-1}x - \int x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$ Ans:(D)

$$\left[\because \frac{d}{dx}\sqrt{1-x^2} = \frac{-x}{\sqrt{1-x^2}}\right]$$

23. Evaluate $\int \frac{1}{\sqrt{e^{2x}-4}}dx = ?$

(A) $\frac{1}{2}\sec^{-1}(\frac{e^x}{2})+C$ (B) $\frac{1}{4}\sec^{-1}(\frac{e^{2x}}{2})+C$ (C) $\frac{1}{2}\sec^{-1}(\frac{e^{2x}}{2})+C$

(D) $\frac{1}{4}\sec^{-1}(\frac{e^x}{4})+C$ (E) $\frac{1}{2}\sec(\frac{e^{2x}}{2})+C$

Sol: $Let: u=e^x \Rightarrow du=e^x dx = udx \Rightarrow \frac{du}{u}=dx$

$$\int \frac{1}{\sqrt{u^2-4}} \times \frac{1}{u}du$$

$$=\int \frac{1}{u\sqrt{u^2-4}}du$$

$$=\frac{1}{2}\cdot\frac{1}{2}\int \frac{1}{\frac{u}{2}\sqrt{\left(\frac{u}{2}\right)^2-1}}du$$

$$=\frac{1}{4}\times 2\sec^{-1}\left|\frac{u}{2}\right|+C$$

$$=\frac{1}{2}\sec^{-1}(\frac{e^x}{2})+C$$

Ans:(A)

24.求橢圓 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,(a>b)$ 繞 x 軸旋轉之旋轉體體積。

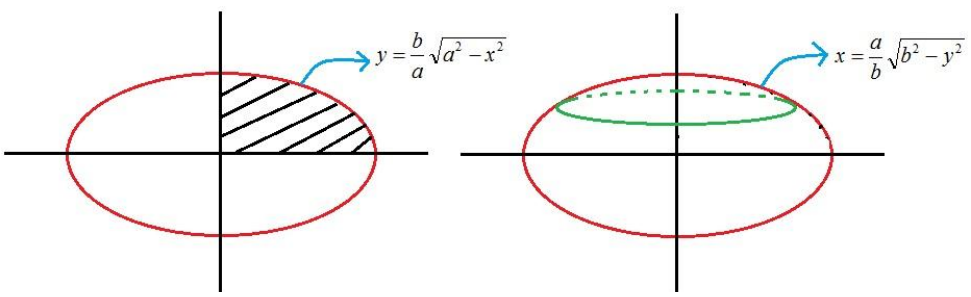
(A) $\frac{3}{4}\pi\cdot a^3$ (B) $\frac{4}{3}\pi\cdot a^2\cdot b$ (C) $\frac{3}{4}\pi\cdot b^3$ (D) $\frac{4}{3}\pi\cdot a\cdot b^2$ (E) $\frac{2}{3}\pi\cdot a^2\cdot b$

Ans:(D)

24.求橢圓 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,(a>b)$ 繞 y 軸旋轉之旋轉體體積。

(A) $\frac{3}{4}\pi\cdot a^3$ (B) $\frac{2}{3}\pi\cdot a^2\cdot b$ (C) $\frac{3}{4}\pi\cdot b^3$ (D) $\frac{4}{3}\pi\cdot a\cdot b^2$ (E) $\frac{4}{3}\pi\cdot a^2\cdot b$

Ans:(E)



$(24)\because \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \Rightarrow \therefore y=\dfrac{b}{a}\sqrt{a^2-x^2}$

$$
\begin{aligned}
V_{y=0} &= 2\int_0^a \pi\cdot r^2 dx \\
&= 2\int_0^a \pi\cdot y^2 dx \\
&= 2\int_0^a \pi\cdot \frac{b^2}{a^2}(a^2-x^2)\cdot dx \\
&= 2\pi\cdot\frac{b^2}{a^2}\left(a^2x-\frac{1}{3}x^3\right)\Bigg|_0^a \\
&= \underline{\underline{\frac{4}{3}\pi\cdot a\cdot b^2}}
\end{aligned}
$$

$(25)\because \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \Rightarrow \therefore y=x=\dfrac{a}{b}\sqrt{b^2-y^2}$

$$
\begin{aligned}
V_{x=0} &= 2\int_0^b \pi\cdot r^2 dx \\
&= 2\int_0^b \pi\cdot x^2 dx \\
&= 2\int_0^b \pi\cdot \frac{a^2}{b^2}(b^2-y^2)\cdot dx \\
&= 2\pi\cdot\frac{a^2}{b^2}\left(b^2y-\frac{1}{3}y^3\right)\Bigg|_0^b \\
&= \underline{\underline{\frac{4}{3}\pi\cdot a^2\cdot b}}
\end{aligned}
$$