國立宜蘭大學 100 學年度微積分競試解答

1. Evaluate $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}=?$

Sol: $Let: y=\sqrt[6]{x} \Leftrightarrow x\to 1 \Leftrightarrow y\to 1$

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}=\lim_{y\to 1}\frac{y^3-1}{y^2-1}=\lim_{y\to 1}\frac{y^2+y+1}{y+1}=\frac{3}{2}$$

2. Evaluate $\lim_{x\to 16}\dfrac{\sqrt{x}-4}{\sqrt{x}-\sqrt[4]{x}-2}=?$

Sol: $Let: y=\sqrt[4]{x} \Leftrightarrow x\to 16 \Leftrightarrow y\to 2$

$$\lim_{x\to 16}\frac{\sqrt{x}-4}{\sqrt{x}-\sqrt[4]{x}-2}=\lim_{y\to 2}\frac{y^2-4}{y^2-y-2}=\lim_{y\to 2}\frac{y+2}{y+1}=\frac{4}{3}$$

3. Find $\lim_{x\to 0}\dfrac{1-\cos x}{5x^2}=?$

Sol:

$$\lim_{x\to 0}\frac{1-\cos x}{5x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2\cdot 4\cdot 5}=\frac{1}{10}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}=\frac{1}{10}$$

4. Find $\lim_{x\to 1}\dfrac{1-x^2}{\sin \pi x}=?$

Sol:

$$\lim_{x\to 1}\frac{1-x^2}{\sin \pi x}=\lim_{x\to 1}\frac{-2x}{\pi\cos \pi x}=\frac{-2}{\pi\times(-1)}=\frac{2}{\pi}$$

5. Evaluate $\dfrac{d}{dx}(9^{3x^2})=?$

Sol: $\dfrac{d}{dx}(9^{3x^2})=\ln 9\cdot 9^{3x^2}\cdot 6x$

6. Evaluate $\dfrac{d}{dx}(x^x)=?$

Sol: $Let: y=(x^x)$

$$\ln y = x\ln x$$

$$\frac{1}{y}\frac{dy}{dx}=\ln x + x\cdot\frac{1}{x}$$

$$\frac{dy}{dx}=y[\ln x+1]=x^x[\ln x+1]$$

7. Evaluate $\dfrac{d}{dx}\left(\dfrac{\arccos x}{\arcsin x}\right)=\dfrac{d}{dx}\left(\dfrac{\cos^{-1}x}{\sin^{-1}x}\right)=?$

Sol:

$$\frac{d}{dx}\left(\frac{\cos^{-1}x}{\sin^{-1}x}\right)=\frac{1}{(\sin^{-1}x)^2}\left(\frac{-1}{\sqrt{1-x^2}}\cdot\sin^{-1}x-\cos^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}\right)$$

$$=\frac{-(\sin^{-1}x+\cos^{-1}x)}{\sqrt{1-x^2}(\sin^{-1}x)^2}$$

8. Assume $f(x)=x^{\frac{1}{x}}$，please find $f'(x)=?$

Sol: $y=x^{\frac{1}{x}}$

$$\ln y=\frac{1}{x}\ln x$$

$$\frac{1}{y}\frac{dy}{dx}=-\frac{1}{x^2}\ln x+\frac{1}{x^2}$$

$$\frac{dy}{dx}=y\left[-\frac{1}{x^2}\ln x+\frac{1}{x^2}\right]=x^{\frac{1}{x}}\left[-\frac{\ln x}{x^2}+\frac{1}{x^2}\right]$$

9. Assume $f(x)=\log_7\left[6x^3-7x^2+5x\right]$，please find $f'(x)=?$

Sol:

$$f(x)=\log_7\left[6x^3-7x^2+5x\right]=\frac{\ln\left[6x^3-7x^2+5x\right]}{\ln 7}$$

$$f'(x)=\frac{1}{\ln 7}\times\frac{18x^2-14x+5}{6x^3-7x^2+5x}$$

10. Evaluate $\int_3^4\dfrac{4x-3}{(2x^2-3x-2)^2}dx=?$

Sol: $Let: u=2x^2-3x-2$

$du=(4x-3)dx$

$$\int_3^4\frac{4x-3}{(2x^2-3x-2)^2}dx=\int_7^{18}\frac{1}{u^2}du=-\frac{1}{u}\bigg|_7^{18}=-\left(\frac{1}{18}-\frac{1}{7}\right)$$

$$=\frac{11}{126}$$

11. If we know $\pounds\{f(t)\}=\int_0^\infty e^{-st}\cdot f(t)\cdot dt$, please evaluate $\pounds\{\sin \alpha t\}=?$

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$$\int_0^\infty e^{-st}\cdot\sin\alpha t\cdot dt$$

$$=-\frac{1}{s}\int_0^\infty(\sin\alpha t)\cdot de^{-st}$$

$$=-\frac{1}{s}\left[(\sin\alpha t)\cdot e^{-st})\Big|_0^\infty-\int_0^\infty e^{-st}\cdot d(\sin\alpha t)\right]$$

$$=-\frac{1}{s}\left[(0-0)-\alpha\int_0^\infty\cos\alpha t\cdot e^{-st}dt\right]$$

$$=-\frac{\alpha}{s^2}\int_0^\infty\cos\alpha t\cdot de^{-st}$$

$$=-\frac{\alpha}{s^2}\left[(\cos\alpha t)\cdot e^{-st})\Big|_0^\infty-\int_0^\infty e^{-st}\cdot d(\cos\alpha t)\right]$$

$$=-\frac{\alpha}{s^2}\left[(0-1)+\alpha\int_0^\infty e^{-st}\cdot(\sin\alpha t)\cdot dt\right]$$

$$=\frac{\alpha}{s^2}-\frac{\alpha^2}{s^2}\int_0^\infty e^{-st}\cdot(\sin\alpha t)\cdot dt$$

$$\Rightarrow Let:\int_0^\infty e^{-st}\cdot(\sin\alpha t)\cdot dt=A$$

$$A=\frac{\alpha}{s^2}-\frac{\alpha^2}{s^2}A$$

$$(1+\frac{\alpha^2}{s^2})A=\frac{\alpha}{s^2}\Rightarrow\frac{s^2+\alpha^2}{s^2}A=\frac{\alpha}{s^2}$$

$$A=\frac{\alpha}{s^2+\alpha^2}=\int_0^\infty e^{-st}\cdot(\sin\alpha t)\cdot dt$$

12. According to 11. please evaluate $\mathcal{L}\{\sin t\cos t\}=?$

$$sol:\mathcal{L}\{\sin t\cos t\}=\mathcal{L}\left\{\frac{1}{2}\sin 2t\right\}=\frac{1}{2}\mathcal{L}\{\sin 2t\}$$

$$=\frac{1}{2}\times\frac{2}{s^2+(2)^2}=\frac{1}{s^2+4}$$

13. Evaluate $\int\sec xdx=?+C$

$$sol:\int\sec xdx=\int\sec x\cdot\frac{\sec x+\tan x}{\sec x+\tan x}dx$$

$$=\ln|\sec x+\tan x|+C$$

14. Evaluate $\int_1^4\frac{2^{\sqrt{x}}}{\sqrt{x}}dx=?$

$$sol:Let:u=\sqrt{x}\Leftrightarrow du=\frac{1}{2\sqrt{x}}dx$$

$$\int_1^4\frac{2^{\sqrt{x}}}{\sqrt{x}}dx=\int_1^2 2^u\cdot 2du=2\times\frac{2^u}{\ln 2}\bigg|_1^2=\frac{4}{\ln 2}$$

15. Evaluate $\int_0^{\ln\sqrt{3}}\frac{e^x}{1+e^{2x}}dx=?$

$$sol:Let:u=e^x\Leftrightarrow du=e^xdx$$

$$\int_0^{\ln\sqrt{3}}\frac{e^x}{1+e^{2x}}dx=\int_1^{\sqrt{3}}\frac{1}{1+u^2}du=\arctan u\Big|_1^{\sqrt{3}}=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$$

16. Evaluate $\int_0^x\frac{(1+\varepsilon x)}{1-x}dx=?$

$$sol:Let:u=1-x\Leftrightarrow du=-dx$$

$$1+\varepsilon x=\varepsilon-\varepsilon u+1$$

$$\int_0^x\frac{(1+\varepsilon x)}{1-x}dx=-\int_1^{1-x}\frac{(1+\varepsilon-\varepsilon u)}{u}du=-\int_1^{1-x}\left[\frac{(1+\varepsilon)}{u}-\varepsilon\right]du$$

$$=\int_1^{1-x}\varepsilon du-\int_1^{1-x}\frac{(1+\varepsilon)}{u}du=\varepsilon u\Big|_1^{1-x}-(1+\varepsilon)\ln u\Big|_1^{1-x}$$

$$=\varepsilon(1-x-1)-(1+\varepsilon)[\ln(1-x)-\ln 1]$$

$$=(1+\varepsilon)\ln\left(\frac{1}{1-x}\right)-\varepsilon x$$

17. Evaluate $\int_0^x\frac{(1+\varepsilon x)}{(1-x)^2}dx=?$

$$sol:Let:u=1-x\Leftrightarrow du=-dx$$

$$1+\varepsilon x=\varepsilon-\varepsilon u+1$$

$$\int_0^x\frac{(1+\varepsilon x)}{(1-x)^2}dx=-\int_1^{1-x}\frac{(1+\varepsilon-\varepsilon u)}{u^2}du$$

$$=\int_1^{1-x}-\frac{(1+\varepsilon)}{u^2}du+\int_1^{1-x}\frac{\varepsilon}{u}du=(1+\varepsilon)\frac{1}{u}\bigg|_1^{1-x}+\varepsilon\ln u\Big|_1^{1-x}$$

$$=(1+\varepsilon)\left[\frac{1}{1-x}-1\right]+\varepsilon[\ln(1-x)-\ln 1]$$

$$=(1+\varepsilon)\frac{x}{1-x}-\varepsilon\ln\left(\frac{1}{1-x}\right)$$

18. Find the area of the region bounded by $x=y^2$ and $x=y+2$.

$$sol:y^2=y+2$$

$$\Rightarrow\begin{cases}y=-1 & \Rightarrow x=1\\ y=2 & \Rightarrow x=4\end{cases}$$

$$A=\int_{-1}^2[(2+y)-y^2]dy=\left(2y+\frac{1}{2}y^2-\frac{1}{3}y^3\right)\bigg|_{-1}^2=\frac{9}{2}$$

19. The region enclosed by $y=3x-x^2$ and $y=0$ is rotated about the line $x=-1$. Find the volume of the resulting solid.

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$sol: 3x - x^2 = 0 \Rightarrow x = 0,3$

$V_{x=-1} = \int_0^3 2\pi r h dx = \int_0^3 2\pi[x-(-1)]f(x)dx$

$= \int_0^3 2\pi[x+1][3x-x^2]dx = 2\pi\int_0^3 [2x^2+3x-x^3]dx$

$= 2\pi\left[-\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{3}{2}x^2\right]_0^3 = \frac{45\pi}{2}$

20. Evaluate $\int_{-10}^{10}\frac{\sin 4x}{\sqrt{1+x^{10}}}dx = ?$

$sol: Let: f(x) = \frac{\sin 4x}{\sqrt{1+x^{10}}}$

$\because f(-x) = \frac{\sin(-4x)}{\sqrt{1+(-x)^{10}}} = -\frac{\sin 4x}{\sqrt{1+x^{10}}} = -f(x)$

$f(x)$ is odd function

$\therefore \int_{-10}^{10}\frac{\sin 4x}{\sqrt{1+x^{10}}}dx = 0$

21. Evaluate $\int_{-\pi}^{\pi}\frac{\cos x}{x\sqrt{1+\pi^5}}dx = ?$

$sol: Let: f(x) = \frac{\cos x}{x\sqrt{1+\pi^5}}$

$\because f(-x) = \frac{\cos(-x)}{(-x)\sqrt{1+\pi^5}} = -\frac{\cos x}{x\sqrt{1+\pi^5}} = -f(x)$

$f(x)$ is odd function

$\therefore \int_{-\pi}^{\pi}\frac{\cos x}{x\sqrt{1+\pi^5}}dx = 0$

22. Evaluate $\int \cos^{-1} x dx = ?$

$sol: \int \cos^{-1} x dx = x\cos^{-1} x - \int x\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$

$= x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}dx$

$= x\cos^{-1} x + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}d(1-x^2)$

$= x\cos^{-1} x - \sqrt{1-x^2} + C$

23. Evaluate $\int \frac{4x^2+20x+25}{16x^3+120x^2+300x+250}dx = ?$

$\int \frac{4x^2+20x+25}{16x^3+120x^2+300x+250}dx = \int \frac{(2x+5)^2}{2(2x+5)^3}dx$

$= \int \frac{1}{2(2x+5)}dx = \int \frac{1}{(4x+10)}dx = \frac{1}{4}\ln|4x+10| + C$

24. Evaluate $\int_{\frac{2}{\sqrt{3}}}^{2} x\sec^{-1} x dx = ?$

$sol: \int_{\frac{2}{\sqrt{3}}}^{2} x\sec^{-1} x dx = \frac{1}{2}\int_{\frac{2}{\sqrt{3}}}^{2} \sec^{-1} x dx^2$

$= \left[\frac{1}{2}x^2\sec^{-1} x\right]_{\frac{2}{\sqrt{3}}}^{2} - \frac{1}{2}\int_{\frac{2}{\sqrt{3}}}^{2} x^2 \frac{1}{x\sqrt{x^2-1}}dx$

$= \frac{5}{9}\pi - \frac{1}{2}\int_{\frac{2}{\sqrt{3}}}^{2}\frac{x}{\sqrt{x^2-1}}dx$

$= \frac{5}{9}\pi - \frac{1}{2}\sqrt{x^2-1}\Big|_{\frac{2}{\sqrt{3}}}^{2}$

$= \frac{5}{9}\pi - \frac{\sqrt{3}}{3}$

25.已知細菌繁殖率與原有細菌數量成正比，假設某一時刻細菌量為 1,000 隻，經過三小時後細菌曾為 8,000 隻，則再經過四小時後細菌量應為多少隻?

以某一時刻為基準

設 t 小時候細菌量為 $f(t)$

由題意知

$f'(t) = k \cdot f(t)$

$\Rightarrow f(t) = y_0 e^{kt}$

$\because f(0) = 1000 = y_0$

$\therefore f(t) = 1000e^{kt}$

$\therefore f(3) = 1000e^{3k} = 8000$

$\Rightarrow e^k = 2$

$f(7) = 1000e^{7k}$

$= 1000(e^k)^7$

$= 1000 \times 2^7$

$= 1000 \times 128 = 128000$