國立宜蘭大學 101 學年度第二學期微積分競試試題解答

1. Evaluate $\lim_{n\to\infty}\left(2^n+4^n\right)^{\frac{1}{n}}=?\quad n\in\mathrm{N}$

(A) 0 (B) 1 (C) 2 (D) 4 (E) Does not exist

Ans:

$$4^n<2^n+4^n<4^n+4^n\Rightarrow\left(4^n\right)^{\frac{1}{n}}<\left(2^n+4^n\right)^{\frac{1}{n}}<\left(4^n+4^n\right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty}(4^n)^{\frac{1}{n}}=4,\lim_{n\to\infty}(2\times4^n)^{\frac{1}{n}}=4\Rightarrow\lim_{n\to\infty}(2^n+4^n)^{\frac{1}{n}}=4$$

$$\begin{cases}\lim_{n\to\infty}(4^n)^{\frac{1}{n}}=4\\ \lim_{n\to\infty}(2\times4^n)^{\frac{1}{n}}=4\end{cases}\Rightarrow\lim_{n\to\infty}(2^n+4^n)^{\frac{1}{n}}=4$$

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2. Evaluate $\lim_{x\to4}\frac{\sqrt{x+12}-x}{x^4-4x^3}$

(A) $-\frac{7}{512}$ (B) $-\frac{7}{412}$ (C) $\frac{7}{412}$ (D) $-\frac{7}{502}$ (E) $\frac{7}{532}$

Ans:

$$\lim_{x\to4}\frac{\sqrt{x+12}-x}{x^4-4x^3}=\lim_{x\to4}\frac{\left(\sqrt{x+12}-x\right)\left(\sqrt{x+12}+x\right)}{\left(x^4-4x^3\right)\left(\sqrt{x+12}+x\right)}=\lim_{x\to4}\frac{x+12-x^2}{\left(x^4-4x^3\right)\left(\sqrt{x+12}+x\right)}$$

$$=\lim_{x\to4}\frac{-(x-4)(x+3)}{\left(x^4-4x^3\right)\left(\sqrt{x+12}+x\right)}=\lim_{x\to4}\frac{-(x-4)(x+3)}{x^3(x-4)\left(\sqrt{x+12}+x\right)}=\lim_{x\to4}\frac{-(x+3)}{x^3\left(\sqrt{x+12}+x\right)}=-\frac{7}{512}$$

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3. Evaluate $\lim_{x\to\frac{\pi}{4}}\left[\tan 2x\cdot\tan\left(\frac{\pi}{4}-x\right)\right]$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) Does not exist

Ans:

Let $t=\frac{\pi}{4}-x$

$\because x\to\frac{\pi}{4}\qquad\therefore t\to0$

$$\lim_{x\to\frac{\pi}{4}}\left[\tan 2x\cdot\tan\left(\frac{\pi}{4}-x\right)\right]=\lim_{t\to0}\left[\tan\left(\frac{\pi}{2}-2t\right)\cdot\tan t\right]=\lim_{t\to0}\left[\cot(2t)\cdot\tan t\right]=\lim_{t\to0}\left[\frac{\cos 2t}{\sin 2t}\times\frac{\sin t}{\cos t}\right]$$

$$=\lim_{t\to0}\left[\frac{\cos 2t}{\cos t}\times\frac{2t}{\sin 2t}\times\frac{\sin t}{t}\times\frac{1}{2}\right]=1\times1\times1\times\frac{1}{2}=\frac{1}{2}$$

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4. Evaluate $\frac{d}{dx}\left[\log_{\sqrt{3}}\tan^{-1}(1+x)\right]$

(A) $\dfrac{2}{\ln 3\cdot\tan^{-1}(1+x)\cdot(x^2+2x+2)}$ (B) $\dfrac{2}{\ln 3\cdot\tan^{-1}(1+x)\cdot(x^2+2x+2)^2}$ (C) $\dfrac{3}{\ln 3\cdot\tan^{-1}(1+x)\cdot(x^2+2x+2)}$

(D) $\dfrac{2}{\ln 3\cdot\tan^{-1}(1-x)\cdot(x^2+2x+2)}$ (E) $\dfrac{\sqrt{3}}{\ln 3\cdot\tan^{-1}(1+x)\cdot(x^2+2x+2)}$

Ans:

$$\frac{d}{dx}\left[\log_{\sqrt{3}}\tan^{-1}(1+x)\right]=\frac{1}{\ln\sqrt{3}\tan^{-1}(1+x)}\cdot\frac{1}{(1+x)^2+1}=\frac{1}{\frac{1}{2}\ln 3\cdot\tan^{-1}(1+x)\cdot(x^2+2x+2)}$$

$$=\frac{2}{\ln 3\cdot\tan^{-1}(1+x)\cdot(x^2+2x+2)}$$

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5. If $y=2^{\tan^{-1}x}+(\ln x)^{\sqrt{x}}$, please find $\dfrac{dy}{dx}$.

(A) $\dfrac{\ln 2\times 2^{\tan^{-1}x}}{1+x^2}+(\ln x)^{\sqrt{x}}\left[\dfrac{\ln(\ln x)}{\sqrt{x}}+\dfrac{\sqrt{x}}{x\ln x}\right]$ (B) $\dfrac{\ln 2\times 2^{\tan^{-1}x}}{1-x^2}+(\ln x)^{\sqrt{x}}\left[\dfrac{\ln(\ln x)}{\sqrt{x}}+\dfrac{\sqrt{x}}{x\ln x}\right]$

(C) $\dfrac{\ln 2\times 2^{\tan^{-1}x}}{1+x^2}+(\ln x)^{\sqrt{x}}\left[\dfrac{\ln(\ln x)}{2\sqrt{x}}+\dfrac{\sqrt{x}}{x\ln x}\right]$ (D) $\dfrac{\ln 2\times 2^{\tan^{-1}x}}{1-x^2}+(\ln x)^{\sqrt{x}}\left[\dfrac{\ln(\ln x)}{2\sqrt{x}}+\dfrac{\sqrt{x}}{x\ln x}\right]$

(E) $\dfrac{\ln 2\times 2^{\tan^{-1}x}}{1-x}+(\ln x)^{\sqrt{x}}\left[\dfrac{\ln(\ln x)}{\sqrt{x}}+\dfrac{\sqrt{x}}{x\ln x}\right]$

Ans:

$$Let: u=(\ln x)^{\sqrt{x}}\Rightarrow \ln u=\sqrt{x}\ln(\ln x)\Rightarrow\frac{1}{u}\cdot\frac{du}{dx}=\frac{1}{2\sqrt{x}}\ln(\ln x)+\sqrt{x}\frac{1}{x\ln x}$$

$$\Rightarrow\frac{du}{dx}=(\ln x)^{\sqrt{x}}\left[\frac{1}{2\sqrt{x}}\ln(\ln x)+\sqrt{x}\frac{1}{x\ln x}\right]$$

$$Let: v=2^{\tan^{-1}x}$$

$$\Rightarrow\frac{dv}{dx}=\ln 2\times 2^{\tan^{-1}x}\cdot\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{dv}{dx}+\frac{du}{dx}=\frac{\ln 2\times 2^{\tan^{-1}x}}{1+x^2}+(\ln x)^{\sqrt{x}}\left[\frac{\ln(\ln x)}{2\sqrt{x}}+\frac{\sqrt{x}}{x\ln x}\right]$$

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6. Considering a function $\sin(xy)=x^2\cos y$, please find the equation of normal line at point $\left(2,\dfrac{\pi}{2}\right)$

(A) $y-\dfrac{\pi}{2}=\dfrac{\pi}{4}(x-2)$ (B) $y-\dfrac{\pi}{2}=-\dfrac{\pi}{4}(x-2)$ (C) $y-\dfrac{\pi}{2}=\dfrac{4}{\pi}(x-2)$ (D) $y-\dfrac{\pi}{2}=-\dfrac{4}{\pi}(x-2)$

(E) $y-\dfrac{\pi}{2}=\dfrac{\pi}{2}(x-2)$

Ans:

$$\sin(xy) = x^2 \cos y$$

$$\Rightarrow \frac{d}{dx}\sin(xy) = \frac{d}{dx}\left(x^2 \cos y\right)$$

$$\Rightarrow \cos(xy)\cdot\left(y + x\frac{dy}{dx}\right) = 2x\cos y + x^2(-\sin y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\cos y - y\cos(xy)}{x\cos(xy) + x^2\sin y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=2\\ y=\frac{\pi}{2}}} = \frac{\pi}{4} \text{...the slope of the tangent line}$$

$$\Rightarrow m = \frac{-1}{\frac{\pi}{4}} = -\frac{4}{\pi} \text{...the slope of the normal line}$$

$$\Rightarrow y - \frac{\pi}{2} = -\frac{4}{\pi}(x-2)$$

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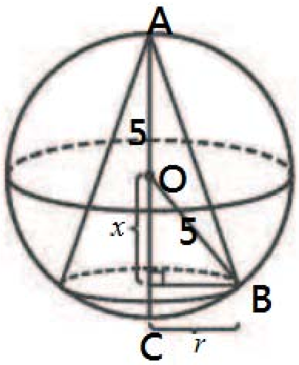
7. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius is 5

(A) $\frac{3125\pi}{27}$ (B) $\frac{1500\pi}{9}$ (C) $\frac{2500\pi}{9}$ (D) $\frac{3750\pi}{27}$ (E) $\frac{4000\pi}{81}$



Ans:

$$\begin{cases} \overline{oc} = x \\ r = \overline{BC} = \sqrt{25 - x^2} \Rightarrow V(x) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(25 - x^2\right)(5 + x) = \frac{1}{3}\pi\left(-x^3 - 5x^2 + 25x + 125\right) \\ h = \overline{AC} = 5 + x \end{cases}$$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi\left(-3x^2 - 10x + 25\right) = -\frac{1}{3}\pi(3x-5)(x+5)$$

$$\Rightarrow \frac{dV}{dx} = 0 \Rightarrow x = \frac{5}{3} \text{ or } -5 \text{ (false)}$$

$$\therefore x = \frac{5}{3}$$

$$\Rightarrow \frac{d^2V}{dx^2} = \frac{1}{3}\pi(-6x - 10) < 0 \Rightarrow V\left(\frac{5}{3}\right) = \frac{4000}{81}\pi$$

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8. If $f(x)$ is continuous on [0 , 2] and differentiable in (0 , 2). Suppose that $f(0) = 2$ and $1 < f'(x) < 2$ for all x in (0 , 2). Find a possible value of $f(2)$

(A)3 (B)4 (C)5 (D)6 (E) 7

Ans:

By The Mean value Theorem

$$f'(x) = \frac{f(2) - f(0)}{2 - 0}$$

$$\Rightarrow 2 < f(2) - f(0) < 4$$

$$\Rightarrow 4 < f(2) < 6$$

$$\Rightarrow \text{take } f(2) = 5$$

【題組 9~10】連續複利

設銀行年利率為 r，每年複利 n 次，本金為 P，

則 t 年後的本利和為 $P_t = P\left(1+\frac{r}{n}\right)^{nt}$

若每年複利一次，則 $P_t = P(1+r)^n$

若每月複利一次，則 $P_t = P\left(1+\frac{r}{12}\right)^{12t}$

若每週複利一次，則 $P_t = P\left(1+\frac{r}{52}\right)^{52t}$

若每日複利一次，則 $P_t = P\left(1+\frac{r}{365}\right)^{365t}$

若每個時刻都在複利，則稱為連續複利，此時 $P_t = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$

9.設彥華在年初向銀行存了 20,000，年利率為 5%，採連續複利計算，則 10 年後彥華可領回多少錢?

(A)54366　(B)45636　(C)64356　(D)35466　(E) 66345

Ans:

$$P_t = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\left[\lim_{n\to\infty} P\left(1+\frac{1}{n/r}\right)^{n/r}\right]^{rt} = P\cdot e^{rt}$$

$$P_{10} = 20{,}000\times e^{0.05\times 20} = 20{,}000\times 2.7183 = 54{,}366$$

10.若彥華想等錢加倍後再領回，至少需要等幾年?

(A)12　(B)14　(C)16　(D)17　(E) 18

Ans:

$$20{,}000\times e^{0.05\times t} \geq 2\times 20{,}000$$

$$\Rightarrow e^{0.05\times t} \geq 2$$

$$\Rightarrow 0.05t \geq \ln 2$$

$$\therefore t \geq \frac{\ln 2}{0.05} \approx 13.86$$

$$\Rightarrow t = 14$$

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11. If $f(x)$ is a continuous function defined in R, and $\int_0^{x^2} f(t)dt = x\sin \pi x$, please find $f(9)$.

(A) -3π　(B) $\frac{2\pi}{3}$　(C) $-\frac{2\pi}{3}$　(D) 3π　(E) $-\frac{1}{2}\pi$

Ans:

$$\int_0^{x^2} f(t)dt = x\sin \pi x$$

$$\Rightarrow \frac{d}{dx}\int_0^{x^2} f(t)dt = \frac{d}{dx}(x\sin \pi x)$$

$$\Rightarrow f(x^2)\times 2x = \sin \pi x + \pi x\cos \pi x$$

$$\Rightarrow x = 3 \Rightarrow f(9)\times 6 = \sin 3\pi + 3\pi\cos 3\pi$$

$$f(9) = -\frac{\pi}{2}$$

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12. Evaluate $\int \cos\sqrt{x}dx$

(A) $\dfrac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{x}}+\mathrm{C}$ (B) $\sqrt{x}\left(\cos\sqrt{x}+\sin\sqrt{x}\right)+\mathrm{C}$ (C) $2\left(\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right)+\mathrm{C}$

(D) $\sqrt{x}\sin\sqrt{x}+\dfrac{\cos\sqrt{x}}{\sqrt{x}}+\mathrm{C}$ (E) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}+\sqrt{x}\cos\sqrt{x}+\mathrm{C}$

Ans:

$$Let: u=\sqrt{x}$$
$$\Rightarrow \frac{du}{dx}=\frac{1}{2\sqrt{x}}$$
$$\Rightarrow dx=2\sqrt{x}du=2udu$$
$$\therefore \int\cos\sqrt{x}dx=\int\cos u\times 2udu=2\int ud\sin u=2\left(u\sin u-\int\sin udu\right)$$
$$=2(u\sin u+\cos u)+C=2\left(\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right)+C$$

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13. Evaluate $\int\frac{1}{4+4x^2+x^4}dx$

(A) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{x^2+2}+C$ (B) $\frac{1}{4}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{4\left(x^2+2\right)}+C$ (C) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{2\left(x^2+2\right)}+C$

(D) $\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{\left(x^2+2\right)}+C$ (E) $\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{4\left(x^2+2\right)}+C$

Ans:

$$4+4x^2+x^4=\left(x^2+2\right)^2$$
$$Let: x=\sqrt{2}\tan\theta$$
$$\Rightarrow \frac{dx}{d\theta}=\sqrt{2}\sec^2\theta$$
$$\Rightarrow dx=\sqrt{2}\sec^2\theta\cdot d\theta$$
$$\therefore \int\frac{1}{4+4x^2+x^4}dx=\int\frac{1}{\left(x^2+2\right)^2}dx=\int\frac{\sqrt{2}\sec^2\theta}{\left(2\tan^2\theta+2\right)^2}d\theta=\frac{\sqrt{2}}{4}\int\frac{1}{\sec^2\theta}d\theta=\frac{\sqrt{2}}{4}\int\cos^2\theta d\theta$$
$$=\frac{\sqrt{2}}{4}\int\left[\frac{1+\cos 2\theta}{2}\right]d\theta=\frac{\sqrt{2}}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C=\frac{\sqrt{2}}{8}(\theta+\sin\theta\cos\theta)+C$$
$$=\frac{\sqrt{2}}{8}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{\sqrt{x^2+2}}\times\frac{\sqrt{2}}{\sqrt{x^2+2}}\right]+C$$
$$=\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{x}{4\left(x^2+2\right)}+C$$

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14. Evaluate $\int \frac{\sec x}{\ln|\sec x+\tan x|}dx =$

(A) $\tan x+C$ (B) $\frac{1}{2}\sec x+C$ (C) $\ln|\sec x+\tan x|+C$ (D) $\ln|\ln|\sec x+\tan x||+C$

(E) $\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|C$

Ans:

$\because \frac{d(\ln|\sec x+\tan x|)}{dx}=\frac{1}{\sec x+\tan x}\cdot(\sec x\cdot\tan x+\sec^2 x)=\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}=\sec x$

$\therefore \sec x dx = d(\ln|\sec x+\tan x|)$

$\int \frac{\sec x}{\ln|\sec x+\tan x|}dx=\int \frac{1}{\ln|\sec x+\tan x|}d(\ln|\sec x+\tan x|)=\ln|\ln|\sec x+\tan x||+C$

15.Evaluate $\int_0^1 \frac{x+1}{x^2+1}dx =$

(A) $1-\ln 2$ (B) $\frac{4\ln 2-\pi}{6}$ (C) $\frac{\pi-\ln 2}{8}$ (D) $\frac{2\ln 2+\pi}{4}$ (E) $\frac{2\ln 2-\pi}{2}$

Ans:

$$\int_0^1 \frac{x+1}{x^2+1}dx=\int_0^1 \frac{x}{x^2+1}dx+\int_0^1 \frac{1}{x^2+1}dx=\left[\frac{1}{2}\ln(x^2+1)+\tan^{-1}x\right]_0^1=\frac{1}{2}\ln 2+\frac{\pi}{4}$$

【題組16~17】

16. Evaluate $\int \frac{\tan^{-1}x}{x^2}dx$

(A) $-\frac{\tan^{-1}x}{x}+\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right)+C$ (B) $-\frac{\sec^{-1}x}{x}-\frac{1}{2}\ln\left(\frac{x}{x^2+1}\right)+C$ (C) $-\frac{\tan^{-1}x}{x^2}+\frac{1}{2}\ln\left(\frac{x^2+1}{x^2}\right)+C$

(D) $\frac{\sec^{-1}x}{x}\cdot\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right)+C$ (E) $\frac{\tan^{-1}x}{x}+\frac{1}{2}\ln\left(\frac{x^2+1}{x^2}\right)+C$

Ans:

$$\int \frac{\tan^{-1}x}{x^2}dx=-\int \tan^{-1}x\, d\frac{1}{x}=-\left[\frac{\tan^{-1}x}{x}-\int\frac{1}{x}d\tan^{-1}x\right]$$

$$=-\frac{\tan^{-1}x}{x}+\int\frac{1}{x}\times\frac{1}{1+x^2}dx=-\frac{\tan^{-1}x}{x}+\int\left[\frac{1}{x}-\frac{x}{1+x^2}\right]dx$$

$$=-\frac{\tan^{-1}x}{x}+\ln|x|-\frac{1}{2}\ln(x^2+1)+C=-\frac{\tan^{-1}x}{x}+\frac{1}{2}\ln x^2-\frac{1}{2}\ln(x^2+1)+C$$

$$=-\frac{\tan^{-1}x}{x}+\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right)+C$$

17. Evaluate $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx$

(A) $\frac{\pi}{2}+\frac{1}{4}\ln 2$ (B) $\frac{\pi}{4}-\frac{1}{2}\ln 5$ (C) $\frac{2\pi}{3}+\frac{1}{5}\ln 5$

(D) $\frac{3\pi}{4}-\ln 2$ (E) $\frac{\pi}{4}+\frac{1}{2}\ln 2$

Ans:

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx = \lim_{t\to\infty}\left[-\frac{\tan^{-1} x}{x}+\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right)\right]_1^t = \left(0+\frac{1}{2}\ln 1\right)-\left(-\frac{\pi}{4}+\frac{1}{2}\ln 2^{-1}\right)=\frac{\pi}{4}+\frac{1}{2}\ln 2$$

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18. Evaluate $\int \ln\left(x^2+2\right)dx$

(A) $x\cdot\ln\left(x^2+2\right)-2x+2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$ (B) $\left(x^2+2\right)\cdot\ln x-2x^2+\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

(C) $x\cdot\ln\left(x^2+2\right)+2x+5\sqrt{2}\sec^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$ (D) $x^2\cdot\ln x^2-2x+2\sqrt{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

(E) $\ln\left(x^2+2\right)-2x^2-2\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}}{x}\right)+C$

Ans:

$Let: u=\ln\left(x^2+2\right)$

$\Rightarrow \frac{du}{dx}=\frac{2x}{x^2+2}$

$\Rightarrow du=\frac{2x}{x^2+2}dx$

$\int \ln\left(x^2+2\right)dx=\int udx = ux-\int xdu$

$=ux-\int x\cdot\frac{2x}{x^2+2}dx$

$=ux-\int \frac{2x^2}{x^2+\left(\sqrt{2}\right)^2}dx$

$=ux-2\int \frac{x^2+\sqrt{2}-\sqrt{2}}{x^2+\left(\sqrt{2}\right)^2}dx$

$=ux-2\int\left[1-\frac{\sqrt{2}}{x^2+\left(\sqrt{2}\right)^2}\right]dx$

$=ux-2\int 1dx+2\int\frac{\sqrt{2}}{x^2+\left(\sqrt{2}\right)^2}dx$

$=x\cdot\ln\left(x^2+2\right)-2x+2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

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19. Please find the area of the region bounded by $y=0$、$x=0$、$x=1$ and the curve represented by parametric:

$x=\ln t$, $y=\dfrac{t+t^{-1}}{2}$.

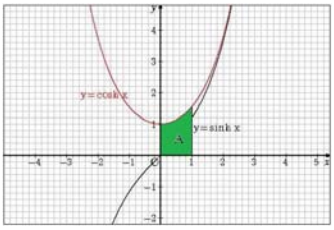
(A) $\cosh 1$ (B) $\sinh 1$ (C) $\coth 1$ (D) $\tanh 1$ (E) $\cosh 2$

Ans:

$\because x=\ln t \quad \therefore t>0$

$\Rightarrow Let: t=e^{x}$

$\Rightarrow y=\dfrac{e^{x}+e^{-x}}{2}=\cosh x$



$\Rightarrow A=\int_0^1 (y-0)dx=\int_0^1 \cosh x dx=\int_0^1\left[\dfrac{e^{x}+e^{-x}}{2}\right]dx=\dfrac{1}{2}\left[\int_0^1 e^{x}dx+\int_0^1 e^{-x}dx\right]=\dfrac{1}{2}\left[e^{x}\Big|_0^1-e^{-x}\Big|_0^1\right]$

$=\dfrac{1}{2}\left[(e-1)-(e^{-1}-1)\right]=\dfrac{1}{2}\left[(e)-(e^{-1})\right]=\sinh 1$

20. The region enclosed According to question19, is rotated about the line $y=0$. Please find the volume of the resulting solid.

(A) $\pi\left[\dfrac{1}{4}+\dfrac{1}{4}\sinh 2\right]$ (B) $\pi\left[\dfrac{1}{4}+\dfrac{1}{2}\sinh 2\right]$ (C) $\pi\left[\dfrac{1}{2}+\dfrac{1}{2}\sinh 2\right]$ (D) $\pi\left[\dfrac{1}{2}+\dfrac{1}{4}\sinh 2\right]$

(E) $\pi\left[\dfrac{1}{4}-\dfrac{1}{2}\sinh 2\right]$

Ans:

$\because x=\ln t \quad \therefore t>0$

$\Rightarrow Let: t=e^{x}$

$\Rightarrow y=\dfrac{e^{x}+e^{-x}}{2}=\cosh x, dy=\sinh x dx$

$\Rightarrow V=\int_0^1 \pi(y-0)^2 dx=\int_0^1 \pi\left(\dfrac{1}{2}+\dfrac{1}{2}\cosh 2x\right)dx=\pi\left[\dfrac{x}{2}+\dfrac{1}{4}\sinh 2x\right]_0^1=\pi\left[\dfrac{1}{2}+\dfrac{1}{4}\sinh 2\right]$

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21. Define that $erf(\eta)=\dfrac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}d\eta$, where $\eta=\dfrac{z}{\sqrt{4kt}}$, k and t both are constant, please evaluate

$\dfrac{d}{dz}(erf(\eta))=?$

(A) $\dfrac{e^{-\eta^2}}{\sqrt{4\pi kt}}$ (B) $\dfrac{2e^{-\eta^2}}{\sqrt{\pi kt}}$ (C) $\dfrac{e^{-\eta^2}}{\sqrt{\pi kt}}$ (D) $\dfrac{4e^{-\eta^2}}{\sqrt{\pi kt}}$ (E)can not solve.

Ans:

$$\frac{d}{dz}\left(erf(\eta)\right)=\frac{d\left(erf(\eta)\right)}{d\eta}\frac{d\eta}{dz}=\frac{d\left(erf(\eta)\right)}{d\eta}\cdot\frac{1}{\sqrt{4kt}}$$

$$\because \frac{d}{dx}\int_{p(x)}^{q(x)} f(t)dt = f(q(x))\cdot q^{/}(x) - f(p(x))\cdot p^{/}(x)$$

$$\therefore \frac{d\left(erf(\eta)\right)}{d\eta}=\frac{2}{\sqrt{\pi}}(e^{-\eta^2}\cdot 1)$$

$$\therefore \frac{d}{dz}\left(erf(\eta)\right)=\frac{e^{-\eta^2}}{\sqrt{\pi kt}}$$

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22. Evaluate $\int_0^{\ln 2} 2e^{-x}\cosh x dx$

(A) $\frac{1}{2}+\ln 2$ (B) $\frac{3}{8}-\ln 2$ (C) $\frac{1}{4}+\ln 2$ (D) $\frac{3}{4}+\ln 2$ (E) $\frac{3}{8}+\ln 2$

Ans:

$$2e^{-x}\cosh x = 2e^{-x}\left[\frac{e^{x}+e^{-x}}{2}\right]=1+e^{-2x}$$

$$\Rightarrow \int_0^{\ln 2} 2e^{-x}\cosh x dx = \int_0^{\ln 2}\left(1+e^{-2x}\right)dx = \left[x-\frac{1}{2}e^{-2x}\right]\Bigg|_0^{\ln 2} = \left[\ln 2-\frac{1}{2}\cdot\frac{1}{4}\right]-\left[0-\frac{1}{2}\right]=\frac{3}{8}+\ln 2$$

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23. Evaluate $\int_0^{\infty} e^{-nt}\cdot\cos\alpha t\cdot dt \quad (n,\alpha \text{ are constant})$

(A) $\frac{n^2}{n^2+\alpha^2}$ (B) $\frac{n}{n^2+\alpha^2}$ (C) $\frac{\alpha^2}{n^2+\alpha^2}$ (D) $\frac{\alpha}{n^2+\alpha^2}$ (E) $\frac{\beta}{n^2+\alpha^2}$

Ans:

$$\int_0^{\infty} e^{-nt}\cdot\cos\alpha t\cdot dt$$

$$=-\frac{1}{n}\int_0^{\infty}(\cos\alpha t)\cdot de^{-nt}$$

$$=-\frac{1}{n}\left[(\cos\alpha t)\cdot e^{-nt})\Big|_0^{\infty}-\int_0^{\infty} e^{-nt}\cdot d(\cos\alpha t)\right]$$

$$=-\frac{1}{n}\left[(0-1)+\alpha\int_0^{\infty}\sin\alpha t\cdot e^{-nt}dt\right]$$

$$=\frac{1}{n}-\frac{1}{n}\left[\alpha\int_0^{\infty}\sin\alpha t\cdot e^{-nt}dt\right]$$

$$=\frac{1}{n}+\frac{\alpha}{n^2}\int_0^{\infty}\sin\alpha t\cdot de^{-nt}$$

$$=\frac{1}{n}+\frac{\alpha}{n^2}\left[\sin\alpha t\cdot e^{-nt}\Big|_0^{\infty}-\int_0^{\infty} e^{-nt}\cdot d\sin\alpha t\right]$$

$$=\frac{1}{n}+\frac{\alpha}{n^2}\left[(0-0)-\alpha\int_0^{\infty} e^{-nt}\cos\alpha t\cdot dt\right]$$

$$=\frac{1}{n}-\frac{\alpha^2}{n^2}\int_0^{\infty} e^{-nt}\cos\alpha t\cdot dt$$

$$\Rightarrow \int_0^{\infty} e^{-nt}\cdot\cos\alpha t\cdot dt=\frac{1}{n}-\frac{\alpha^2}{n^2}\int_0^{\infty} e^{-nt}\cos\alpha t\cdot dt$$

$$\Rightarrow \left(1+\frac{\alpha^2}{n^2}\right)\int_0^{\infty} e^{-nt}\cos\alpha t\cdot dt=\frac{1}{n}$$

$$\Rightarrow \int_0^{\infty} e^{-nt}\cos\alpha t\cdot dt=\frac{1}{n}\times\left(1+\frac{\alpha^2}{n^2}\right)^{-1}=\frac{1}{n}\times\left(\frac{n^2}{n^2+\alpha^2}\right)=\frac{n}{n^2+\alpha^2}$$

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24. Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plan region about the y-axis.

$$y=\begin{cases}\dfrac{\sin x}{x}, x>0\\ 1, x=0\end{cases},\quad y=0,\quad x=0,\quad x=\pi$$

(A) 8π (B) 7π (C) 6π (D) 5π (E) 4π

Ans:

$$\begin{cases}p(x)=x\\ h(x)=\dfrac{\sin x}{x}\end{cases}$$

$$V=2\pi\int_a^b p(x)h(x)dx=2\pi\int_0^{\pi} x\cdot\frac{\sin x}{x}dx=2\pi\int_0^{\pi}\sin x dx=-2\pi[\cos x]_0^{\pi}=4\pi$$

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25. Evaluate $\int_{-7}^{7}\dfrac{\sin(5x)}{\cos x\sqrt{1+x^2+x^4+x^6}}\cdot dx$

(A)7 (B)0 (C)0.49 (D)-7 (E)14

Ans:

$$Let: f(x)=\frac{\sin(5x)}{\cos x\sqrt{1+x^2+x^4+x^6}}$$

$$\because f(-x)=\frac{\sin(-5x)}{\cos(-x)\sqrt{1+(-x)^2+(-x)^4+(-x)^6}}=-\frac{\sin(5x)}{\cos x\sqrt{1+x^2+x^4+x^6}}=-f(x)$$

$\therefore f(x)$ is odd function

$$\Rightarrow \int_{-7}^{7}\frac{\sin(5x)}{\cos x\sqrt{1+x^2+x^4+x^6}}\cdot dx=0$$

Appendix

1. $e\approx 2.7183$

2. $\begin{cases}\ln 2\approx 0.6931\\ \ln 3\approx 1.0986\\ \ln 4\approx 1.3863\\ \ln 5\approx 1.6094\end{cases}$ 3. $\begin{cases}\log 2\approx 0.3010\\ \log 3\approx 0.4771\\ \log 4\approx 0.6021\\ \log 5\approx 0.6990\end{cases}$

4. $\sinh x=\dfrac{e^x-e^{-x}}{2}$ 5. $\cosh x=\dfrac{e^x+e^{-x}}{2}$ 6. $\tanh x=\dfrac{\sinh x}{\cosh x}$

7. If $\sin^{-1}x=y \Rightarrow \sin y=x$

8. $\pi\approx 3.1415926$

9. cone:錐體

10. compounding:複利

11. represented:代表

12. parametric:參數