1 0 4 年 度 第 二 次 微 積 分 競 試　　答 案

1. $\ln(x+\sqrt{x^2+1})$	2. $\frac{15}{4}$
3. $y=1$	4. $\frac{1}{9}$
5. $x^{\sin x}(\cos x\ln x+\frac{\sin x}{x})$	6. 3
7. $y=\frac{1}{e}x$	8. $(2,0),\quad(\frac{3}{2},-\frac{1}{16})$
9. $\sec^2(\tan\theta)\cdot\sec^2\theta$	10. $\frac{1}{2}+\frac{\sqrt{3}}{2}$
11. $1-\frac{\pi}{4}$	12 $\frac{\pi}{8}$
13. $\frac{2}{3}$	14 $-2x\sin xy-x^2y\cos xy$
15. -8	16. 1
17. $3\sqrt{2}$	18. $\frac{8}{3}-\frac{4}{3}\sqrt{2}$
19. 1	20. $\frac{4}{3}$
21. $\frac{2}{27}(10\sqrt{10}-1)$	22. 34
23. $\frac{\pi}{2}$	

1. 令 $y=\frac{e^x-e^{-x}}{2}$， $2y=e^x-e^{-x}$， $2ye^x=e^{2x}-1$， $(e^x)^2-2ye^x-1=0$

$e^x=y\pm\sqrt{y^2+1}$ (負不合)， $e^x=y+\sqrt{y^2+1}$， $x=\ln(y+\sqrt{y^2+1})$

$\therefore\ \sinh^{-1}x=\ln(x+\sqrt{x^2+1})$

2. $\lim\limits_{x\to 0}\frac{\tan 3x\sin 5x}{x\sin 4x}=\lim\limits_{x\to 0}\frac{3(\sin 3x)5(\sin 5x)(4x)}{(3x)(\cos 3x)(5x)4(\sin 4x)}=\frac{15}{4}$

$\because \lim\limits_{x\to 0}\frac{\sin 3x}{3x}=1$， $\lim\limits_{x\to 0}\frac{1}{\cos 3x}=1$， $\lim\limits_{x\to 0}\frac{\sin 5x}{5x}=1$， $\lim\limits_{x\to 0}\frac{4x}{\sin 4x}=1$

3. $y=x\tan x$，令 $z=\frac{1}{x}$， $x\to\infty$， $z\to 0^+$， $\lim\limits_{x\to\infty}x\tan\frac{1}{x}=\lim\limits_{z\to 0^+}\frac{\tan z}{z}=\lim\limits_{z\to 0^+}\frac{\sec^2 z}{1}=1$

$x\to-\infty$， $z\to 0^-$， $\lim\limits_{x\to-\infty}x\tan\frac{1}{x}=1$　　$\therefore$ $y=1$ 為水平漸近線

4. $f(a)=b$， $f^{-1}(b)=a$，則 $\left.\frac{d}{dx}f^{-1}(x)\right|_{x=b}=\frac{1}{\left.f'(x)\right|_{x=a}}$

$f(x)=2x^3+3x-3$，令 $2x^3+3x-3=2$ $\Rightarrow$ $x=1$， $f(1)=2$， $f^{-1}(2)=1$

$\therefore\ g'(2)=\frac{1}{\left.f'(x)\right|_{x=1}}=\frac{1}{\left.(6x^2+3)\right|_{x=1}}=\frac{1}{9}$

5. $y=x^{\sin x}$， $\ln y=\sin x\ln x$， $\frac{1}{y}\frac{dy}{dx}=\cos x\ln x+\frac{\sin x}{x}$， $\frac{dy}{dx}=y(\cos x\ln x+\frac{\sin x}{x})$

$\therefore\ \frac{dy}{dx}=x^{\sin x}(\cos x\ln x+\frac{\sin x}{x})$

6. $f'(0)=1$ $\Rightarrow$ $\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=1$ $\Rightarrow$ $\lim\limits_{h\to 0}\frac{f(5h)-f(0)}{5h}=1$， $\lim\limits_{h\to 0}\frac{f(2h)-f(0)}{2h}=1$

$\lim\limits_{h\to 0}\frac{f(5h)-f(2h)}{h}=\lim\limits_{h\to 0}\frac{f(5h)-f(0)-f(2h)+f(0)}{h}$

$=5\lim\limits_{h\to 0}\frac{f(5h)-f(0)}{5h}-2\lim\limits_{h\to 0}\frac{f(2h)-f(0)}{2h}=5-2=3$

7. $y=\ln x$，令切點為 $(a,\ln a)$， $y'=\frac{1}{x}$， $m=\frac{1}{a}$， $m=\frac{\ln a-0}{a-0}=\frac{1}{a}$ $\Rightarrow$ $\ln a=1$， $a=e$

$\therefore\ y=\frac{1}{e}x$

8. $f(x)=(x-2)^3(x-1)$， $f'(x)=(x-2)^2(4x-5)$， $f''(x)=6(x-2)(2x-3)$

令 $f''(x)=0$ $\Rightarrow$ $x=2$， $x=\frac{3}{2}$ $\Rightarrow$ $(2,0)$， $(\frac{3}{2},-\frac{1}{16})$

9. 令 $z=\tan\theta$， $\frac{d}{dz}\int_0^z \sec^2 x dx=\sec^2 z$

$$\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 x dx=\frac{d}{dz}\int_0^z \sec^2 x dx\frac{dz}{d\theta}=\sec^2 z\sec^2\theta=\sec^2(\tan\theta)\sec^2\theta$$

10 $\int_0^{\frac{\pi}{3}}(\sin x+\cos)dx=(-\cos x+\sin x)\Big|_0^{\frac{\pi}{3}}=\frac{1}{2}+\frac{\sqrt{3}}{2}$

11 $\int_0^{\frac{\pi}{4}}\tan^2 x dx=\int_0^{\frac{\pi}{4}}(\sec^2 x-1)dx=\int_0^{\frac{\pi}{4}}\sec^2 x dx-\int_0^{\frac{\pi}{4}}dx=\tan x\Big|_0^{\frac{\pi}{4}}-(\frac{\pi}{4}-0)=1-\frac{\pi}{4}$

12 $\int_0^2\frac{1}{4+x^2}dx=\frac{1}{2}\int_0^2\frac{1}{1+(\frac{x}{2})^2}d(\frac{x}{2})=\frac{1}{2}\tan^{-1}(\frac{x}{2})\Big|_0^2=\frac{1}{2}(\tan^{-1}1-\tan^{-1}0)=\frac{\pi}{8}$

13 $\int_{-1}^0\sqrt{y+1}dy=\int_0^1\sqrt{z}dz=\frac{2}{3}$

14 $f(x,y)=x\cos xy$， $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}[-x\sin(xy)\cdot x]=-2x\sin xy-x^2y\cos xy$

15 $\int_{-1}^1\int_{-2}^2(x^2-y^2)dydx=\int_{-1}^1(x^2y-\frac{1}{3}y^3)\Big|_{-2}^2 dx=2\int_{-1}^1(2x^2-\frac{8}{3})dx=-8$

16 $f(x)=\frac{ax^3+bx^2+cx+d}{x^2+x-2}$， $\because \lim\limits_{x\to\infty}f(x)=1 \Rightarrow a=0,\quad b=1$

$\because \lim\limits_{x\to 1}f(x)=0 \Rightarrow x^2+cx+d$ 有 $(x-1)^2$ 的因式 $\Rightarrow d=1$

17 $x+y+z=0,\quad x^2+2z^2=1 \Rightarrow$ 令 $x=\cos t,\quad z=\frac{\sin t}{\sqrt{2}},\quad y=-\cos t-\frac{\sin t}{\sqrt{2}}$

$$f(x,y,z)=3x-y-3z=3\cos t+\cos t+\frac{\sin t}{\sqrt{2}}-3\frac{\sin t}{\sqrt{2}}=4\cos t-\sqrt{2}\sin t$$

$$-\sqrt{18}\le f\le\sqrt{18}$$

18 $\int_0^1\frac{dx}{\sqrt{1+\sqrt{x}}}=\int_0^1\frac{2udu}{\sqrt{1+u}}=\int_1^2\frac{2(z-1)}{\sqrt{z}}dz=2\int_1^2 z^{\frac{1}{2}}dz-2\int_1^2 z^{-\frac{1}{2}}dz=\frac{8}{3}-\frac{4}{3}\sqrt{2}$

$u=\sqrt{x}$， $du=\frac{1}{2}\frac{1}{\sqrt{x}}dx,\quad dx=2udu$， $x=0,\quad u=0;\quad x=1,\quad u=1$

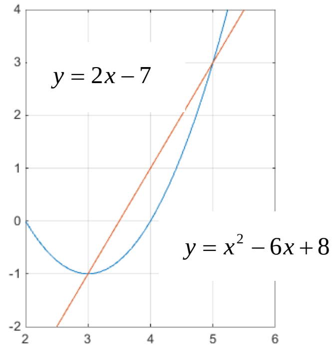
$z=1+u,\quad u=z-1,\quad dz=du$， $u=0,\quad z=1;\quad u=1,\quad z=2$

19 $\int udv=uv-\int vdu$ 令 $u=\ln x,\quad dv=x^{-2}dx,\quad du=x^{-1}dx,\quad v=-x^{-1}$

$$\int\frac{\ln x}{x^2}dx=-\frac{\ln x}{x}+\int\frac{dx}{x^2}=-\frac{\ln x}{x}-\frac{1}{x}+C$$

$$\int_1^a \frac{\ln x}{x^2}dx = (-\frac{\ln x}{x} - \frac{1}{x})\Big|_1^a = -\frac{\ln a}{a} - \frac{1}{a} + 0 + 1, \quad a \to \infty, \quad \frac{1}{a} \to 0, \quad \frac{\ln a}{a} \to \frac{\frac{1}{a}}{1} \to 0$$

$$\therefore \quad \int_1^\infty \frac{\ln x}{x^2}dx = 1$$



20 $y = x^2 - 6x + 8 = (x-3)^2 - 1, \quad y = 2x - 7$

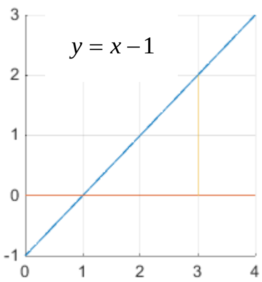
令 $x^2 - 6x + 8 = 2x - 7 \quad \Rightarrow \quad x = 3, \quad x = 5$

$$A = \int_3^5 [(2x-7) - (x^2 - 6x + 8)]dx = \int_3^5 (8x - 15 - x^2)dx = \frac{4}{3}$$

21 $y = (\frac{x}{2})^{\frac{2}{3}}, \quad x = 2y^{\frac{3}{2}}, \quad \frac{dx}{dy} = 3y^{\frac{1}{2}}$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dy})^2 + 1}\, dy = \sqrt{9y+1}\, dy$$

$$s = \int_0^1 \sqrt{9y+1}\, dy = \frac{2}{27}(10\sqrt{10} - 1)$$



22 $R: \quad 0 \le x \le 3, \quad 0 \le y \le x - 1$

$$\iint_R (5x + 8y)dxdy = \int_1^3 \int_0^{x-1} (5x + 8y)dydx = 34$$

23 $x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2$

$$\iint_\Omega (x^2 + y^2)dxdy = \int_0^1 \int_0^{2\pi} r^2 r d\theta dr = \int_0^1 \int_0^{2\pi} r^3 d\theta dr = 2\pi \int_0^1 r^3 dr = \frac{\pi}{2}$$