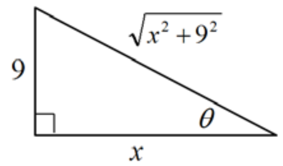
國立宜蘭大學 106 年度第二次微積分競試 解答

1.	$\frac{9}{\sqrt{x^2+81}}$	2.	$-\sqrt{2}$
3.	1	4.	$z=\pm 2$
5.	$-\infty$	6.	$\frac{3x^2\sec x^3\tan x^3}{2\sqrt{\sec x^3}}$
7.	$\frac{2x}{3y^2+2y-5}$	8.	$x^{\sin x}(\cos x\ln x+\frac{\sin x}{x})$
9.	$3x^2\sin^{-1}x+\frac{x^3}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}$	10.	$(0,0),\quad(2,-16)$
11.	$\frac{1}{4}e^{2x}(2x^2-2x+1)+C$	12.	$\ln\left\|\frac{x-1}{x+4}\right\|+C$
13.	$\frac{\pi}{6}$	14.	π
15.	-2	16.	(a) (b) (d)
17.	$y=x+2,\quad y=x-2$	18.	$1-\frac{\sqrt{3}}{6}\pi$
19.	$\ln(\sqrt{2}+1)$	20.	$\frac{13}{9}$
21.	0	22.	$6-18\pi$
23.	$72\sqrt{2}\pi$	24.	$\frac{27}{2}\pi$

1. $x>0,\quad \tan^{-1}\frac{9}{x}=\theta,\quad \tan\theta=\frac{9}{x},\quad \Rightarrow\quad \sin\theta=\frac{9}{\sqrt{x^2+81}}$



2. $\lim\limits_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}=\lim\limits_{x\to\frac{\pi}{4}}\frac{\frac{\cos x-\sin x}{\cos x}}{\sin x-\cos x}=-\lim\limits_{x\to\frac{\pi}{4}}\frac{1}{\cos x}=-\sqrt{2}$

3. $\lim\limits_{x\to 0}\frac{1-e^{-x}}{e^x-1}=\lim\limits_{x\to 0}\frac{\frac{e^x-1}{e^x}}{e^x-1}=\lim\limits_{x\to 0}\frac{1}{e^x}=1$

4. $f(z)=\ln(z^2-4),\quad \lim\limits_{z\to 2^+}\ln(z^2-4)=-\infty,\quad \lim\limits_{z\to -2^-}\ln(z^2-4)=-\infty,$ 鉛直漸近線為 $z=\pm 2$

5. $\lim\limits_{x\to(\frac{\pi}{2})^-}\ln|\cos x|=-\infty$

6. $y=\sqrt{\sec x^3},\quad \frac{dy}{dx}=\frac{\sec x^3\tan x^3}{2\sqrt{\sec x^3}}\cdot 3x^2=\frac{3x^2\sec x^3\tan x^3}{2\sqrt{\sec x^3}}$

7. $y^3+y^2-5y-x^2=-4,\qquad 3y^2\frac{dy}{dx}+2y\frac{dy}{dx}-5\frac{dy}{dx}=2x,\qquad \frac{dy}{dx}=\frac{2x}{3y^2+2y-5}$

8. $y=x^{\sin x},\qquad \ln y=\sin x\ln x,\qquad \frac{1}{y}\frac{dy}{dx}=\cos x\ln x+\frac{\sin x}{x},\qquad \frac{dy}{dx}=x^{\sin x}(\cos x\ln x+\frac{\sin x}{x})$

9. $y=x^3\sin^{-1}x+\cos^{-1}\sqrt{x},\quad \frac{dy}{dx}=3x^2\sin^{-1}x+x^3\cdot\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx}=3x^2\sin^{-1}x+\frac{x^3}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}$

10. $y=x^4-4x^3,\quad \frac{dy}{dx}=4x^3-12x^2,\quad \frac{d^2y}{dx^2}=12x^2-24x,$ 令 $\frac{d^2y}{dx^2}=0$，得 $x=0,\quad x=2$

反曲點 $(0,0),\quad (2,-16)$

11. $\int udv=uv-\int vdu,\quad u=x^2,\quad dv=e^{2x}dx,\quad du=2xdx,\quad v=\frac{1}{2}e^{2x}$

$\int x^2e^{2x}dx=\frac{1}{2}x^2e^{2x}-\int xe^{2x}dx,\qquad u=x,\quad dv=e^{2x}dx,\quad du=dx,\quad v=\frac{1}{2}e^{2x}$

$=\frac{1}{2}x^2e^{2x}-[\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}dx]=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+C=\frac{1}{4}e^{2x}(2x^2-2x+1)+C$

12. $\frac{5}{x^2+3x-4}=\frac{5}{(x+4)(x-1)}=\frac{a}{x+4}+\frac{b}{x-1},\quad a(x-1)+b(x+4)=5,\quad a=-1,\quad b=1$

$\int\frac{5}{x^2+3x-4}dx=\int(\frac{1}{x-1}-\frac{1}{x+4})dx=\ln|x-1|-\ln|x+4|+C=\ln\left|\frac{x-1}{x+4}\right|+C$

13. $\int_0^{\frac{\sqrt{3}}{2}}\frac{1}{1+4x^2}dx$，令 $z=2x,\quad dz=2dx,\quad x=0,\quad z=0;\quad x=\frac{\sqrt{3}}{2},\quad z=\sqrt{3}$

$\int_0^{\frac{\sqrt{3}}{2}}\frac{1}{1+4x^2}dx=\frac{1}{2}\int_0^{\sqrt{3}}\frac{dz}{1+z^2}=\frac{1}{2}\Big[\tan^{-1}z\Big]_0^{\sqrt{3}}=\frac{1}{2}(\tan^{-1}\sqrt{3}-\tan^{-1}0)=\frac{1}{2}\cdot\frac{\pi}{3}=\frac{\pi}{6}$

14. $\int_0^{\infty}\frac{dx}{\sqrt{x}(x+1)}$，　令 $z=\sqrt{x}$，$dz=\frac{dx}{2\sqrt{x}}$

$$\int_0^{\infty}\frac{dx}{\sqrt{x}(x+1)}=2\int_0^{\infty}\frac{dz}{(z^2+1)}=2\lim_{b\to\infty}\left[\tan^{-1}z\right]_0^b=2\cdot\frac{\pi}{2}=\pi$$

15. $f(x,y)=x^2-3y^2$，$\vec{u}=\frac{4}{5}\vec{i}+\frac{3}{5}\vec{j}$

$$D_{\vec{u}}f(x,y)=(\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j})\cdot\vec{u}=(2x\vec{i}-6y\vec{j})(\frac{4}{5}\vec{i}+\frac{3}{5}\vec{j})=\frac{8x}{5}-\frac{18y}{5}$$

$$D_{\vec{u}}f(1,1)=\frac{8}{5}-\frac{18}{5}=-\frac{10}{5}=-2$$

16. (a)(b)(d)

17. 設 $y=mx+b$ 為 $y=\frac{x(|x|+2)}{\sqrt{x^2-1}}$ 的斜漸近線，則 $\lim_{x\to\pm\infty}(\frac{x(|x|+2)}{\sqrt{x^2-1}}-mx-b)=0$

$$\therefore\quad b=\lim_{x\to\pm\infty}(\frac{x(|x|+2)}{\sqrt{x^2-1}}-mx),\qquad m=\lim_{x\to\pm\infty}\frac{x(|x|+2)}{x\sqrt{x^2-1}}$$

(1) $x>0$

$$m=\lim_{x\to\infty}\frac{x+2}{\sqrt{x^2-1}}=1,$$

$$b=\lim_{x\to\infty}(\frac{x(x+2)}{\sqrt{x^2-1}}-x)=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}(x+2-\sqrt{x^2-1})$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}(x+2-\sqrt{x^2-1})(\frac{x+2+\sqrt{x^2-1}}{x+2+\sqrt{x^2-1}})=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}(\frac{x^2+4x+4-x^2+1}{x+2+\sqrt{x^2-1}})$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}\lim_{x\to\infty}\frac{4x+5}{x+2+\sqrt{x^2-1}}=(1)(\frac{4}{1+1})=2,$$

$y=x+2$ 為斜漸近線

(2) $x<0$

$$m=\lim_{x\to-\infty}\frac{-x+2}{\sqrt{x^2-1}}=\lim_{z\to\infty}\frac{z+2}{\sqrt{z^2-1}}=1,$$ 其中 $z=-x$，$z^2=x^2$

$$b=\lim_{x\to-\infty}(\frac{x(-x+2)}{\sqrt{x^2-1}}-x)=\lim_{x\to-\infty}\frac{-x}{\sqrt{x^2-1}}(x-2+\sqrt{x^2-1})$$

$$=\lim_{x\to-\infty}\frac{-x}{\sqrt{x^2-1}}(x-2+\sqrt{x^2-1})(\frac{x-2-\sqrt{x^2-1}}{x-2-\sqrt{x^2-1}})=\lim_{x\to-\infty}\frac{-x}{\sqrt{x^2-1}}(\frac{x^2-4x+4-x^2+1}{x-2-\sqrt{x^2-1}})$$

$$=\lim_{x\to-\infty}\frac{-x}{\sqrt{x^2-1}}\lim_{x\to-\infty}\frac{-4x+5}{x-2-\sqrt{x^2-1}}=\lim_{z\to\infty}\frac{z}{\sqrt{z^2-1}}\lim_{z\to\infty}\frac{4z+5}{-z-2-\sqrt{z^2-1}}=(1)(\frac{4}{-1-1})=-2$$

$y=x-2$ 為斜漸近線

18. $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx$, $\quad 1+\tan^2\theta = \sec^2\theta, \quad \tan^2\theta = \sec^2\theta - 1$

$\sqrt{x^2-3} = \sqrt{3}\sqrt{\frac{x^2}{3}-1}$, 令 $\sec\theta = \frac{x}{\sqrt{3}}, \quad x = \sqrt{3}\sec\theta, \quad dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-3} = \sqrt{3}\tan\theta, \quad x=2,\ \sec\theta = \frac{2}{\sqrt{3}},\ \theta = \frac{\pi}{6}; \quad x=\sqrt{3},\ \sec\theta = 1,\ \theta = 0;$

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta\, d\theta = \sqrt{3}\int_0^{\frac{\pi}{6}} \tan^2\theta\, d\theta = \sqrt{3}\int_0^{\frac{\pi}{6}} (\sec^2\theta - 1)\, d\theta$$

$$= \sqrt{3}[\tan\theta - \theta]_0^{\pi/6} = \sqrt{3}(\frac{1}{\sqrt{3}} - \frac{\pi}{6}) = 1 - \frac{\sqrt{3}}{6}\pi$$

19. $y = \ln(\cos x), \quad x \in [0, \frac{\pi}{4}]$, $\quad \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \sqrt{1+\tan^2 x}\, dx = \sec x\, dx$$

$$S = \int ds = \int_0^{\frac{\pi}{4}} \sec x\, dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4} = \ln(\sqrt{2}+1) - \ln(1) = \ln(\sqrt{2}+1)$$

20. $\vec{N} = 8x\vec{i} - 2y\vec{j} + \vec{k}, \quad \vec{N}(1,-2,0) = 8\vec{i} + 4\vec{j} + \vec{k}$,

$$\vec{n} = \pm\frac{8\vec{i} + 4\vec{j} + \vec{k}}{\sqrt{8^2+4^2+1}} = \pm(\frac{8}{9}\vec{i} + \frac{4}{9}\vec{j} + \frac{1}{9}\vec{k}) = a\vec{i} + b\vec{j} + c\vec{k}, \qquad |a|+|b|+|c| = \frac{13}{9}$$

21. $S_1 : F_1(x,y,z) = 3x^2 - 2y^2 - z^2 = 0$, $\quad \vec{N}_1 = \nabla F_1(x,y,z) = 6x\vec{i} - 4y\vec{j} - 2z\vec{k}$

$S_2 : F_2(x,y,z) = x + y + z - 3 = 0$, $\quad \vec{N}_2 = \nabla F_2(x,y,z) = \vec{i} + \vec{j} + \vec{k}$

$\vec{N}_1(1,1,1) = 6\vec{i} - 4\vec{j} - 2\vec{k}, \quad \vec{N}_2(1,1,1) = \vec{i} + \vec{j} + \vec{k}$

As $\vec{T}$ is orthogonal to both $\vec{N}_1$ and $\vec{N}_2$, $\vec{T}$ must be parallel (or antiparallel) with $\vec{\tau}$

where $\vec{\tau} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & -2 \\ 1 & 1 & 1 \end{vmatrix} = -2\vec{i} - 8\vec{j} + 10\vec{k}$, $\quad \vec{T} = a\vec{i} + b\vec{j} + c\vec{k}, \quad a > 0$

$$\vec{T} = \frac{2\vec{i} + 8\vec{j} - 10\vec{k}}{\sqrt{4+64+100}} \quad \Rightarrow \quad a = \frac{2}{\sqrt{168}}, \quad b = \frac{8}{\sqrt{168}}, \quad c = \frac{-10}{\sqrt{168}}, \qquad a+b+c = 0$$

22. $w = w(x,y) = xy\cos y, \quad x = 3e^{2t}, \quad y = 3\pi - 2t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = y\cos y \cdot 6e^{2t} + (x\cos y - xy\sin y)(-2)$$

$t = 0, \quad x(0) = 3, \quad y(0) = 3\pi$,

$$\left.\frac{dw}{dt}\right|_{t=0} = 3\pi\cos(3\pi)\cdot 6 + (3\cos(3\pi) - 3(3\pi)\sin(3\pi))(-2) = -18\pi + 6 = 6 - 18\pi$$

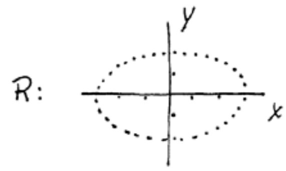
23. Taking spherical coordinates

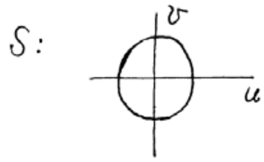
$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$$V = \iiint dv = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=6}\int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} (\rho^2\sin\phi)d\phi\, d\rho\, d\theta = \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\rho=0}^{\rho=6}\rho^2 d\rho \int_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} \sin\phi\, d\phi$$

$$= (2\pi)\left(\left.\frac{\rho^3}{3}\right|_0^6\right)(-1)\left(\cos\phi\Big|_{\pi/4}^{\pi/2}\right) = (2\pi)(72)(-1)(0 - \frac{\sqrt{2}}{2}) = 72\sqrt{2}\pi$$

24. $\iint_R x^2 dA, \qquad R: \ (\frac{x}{3})^2 + (\frac{y}{2})^2 \le 1$





Let $x = 3u, \quad y = 2v, \qquad S: \quad u^2 + v^2 \le 1$

coordinate transformation from (x,y) to (u,v),

$x = g(u,v),\ y = h(u,v)$

$$\iint_R f(x,y)dA = \iint_R f(x,y)dxdy = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

$$\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_S (3u)^2 \cdot 6 \cdot dudv = 54\iint_S u^2 dudv$$

Taking coordinate transformation again,

Let $u = r\cos\theta, \quad v = r\sin\theta$

$$54\iint_S u^2 dudv = 54\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\, rdrd\theta = 54\left(\left.\frac{r^4}{4}\right|_0^1\right)\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{54}{4}\int_0^{2\pi}(\frac{1+\cos 2\theta}{2})d\theta$$

$$= \frac{27}{2}\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{27}{2}\pi$$