國立宜蘭大學 107 年度微積分競試 解答

1. $(-\infty,\frac{1}{3})\cup[\frac{4}{11},\infty)$	2. $\frac{1+5\ln 3}{-2+\ln 3}$
3. 不存在	4. $(A,B)=(7,4)$
5. 2	6. $\frac{-2y}{x(8y^2+1)}$
7. $\frac{(2x^2+2x-1)\sqrt{x-1}}{\sqrt{(x+1)^3}}$	8. $(\frac{3}{2},\frac{\sqrt{6}}{2})$
9. 上凹：$(-\infty,\frac{3}{2}),(2,\infty)$，下凹：$(\frac{3}{2},2)$	10. 相對極大$(2,4e^{-2})$，相對極小$(0,0)$
11. $\frac{(\ln x)^3}{3}+C$	12. $\sec^2\frac{x}{2}+C$ 或 $\tan^2\frac{x}{2}+C$
13. $\frac{9\pi}{20}$	14. $\frac{1}{2}\ln\frac{8}{5}-\frac{\pi}{4}+\tan^{-1}2$
15. $\frac{4698\pi}{5}$	16. 4
17. $\sqrt{17}$	18. 147
19. 6π	20. $x-y-2z+6=0$
21. $\frac{\partial w}{\partial t}=\frac{-2st^2-2s^3}{t^2}$	22. $\frac{\partial z}{\partial y}=\frac{2x^2y-3z}{3x^2+6z^2+3y}$
23. $4\sqrt{2}\pi$	24. $\frac{9\pi}{4}$

1. $f(x)=\dfrac{x}{3x-1}$, $g(x)=\sqrt{4-x}$, $g\circ f(x)=\sqrt{4-\dfrac{x}{3x-1}}=\sqrt{\dfrac{11x-4}{3x-1}}$

$\dfrac{11x-4}{3x-1}\geq 0,\quad x\neq\dfrac{1}{3}\quad\Rightarrow\quad x<\dfrac{1}{3}$ 或 $x\geq\dfrac{4}{11}$

2. $e^{2x+1}=3^{x-5}$，取對數，$2x+1=(x-5)\ln 3\quad\Rightarrow\quad x=\dfrac{1+5\ln 3}{-2+\ln 3}$

3. $\lim\limits_{x\to 1^+}\dfrac{x^2-1}{|x-1|}=\lim\limits_{x\to 1^+}\dfrac{x^2-1}{x-1}=\lim\limits_{x\to 1^+}(x+1)=2$, $\lim\limits_{x\to 1^-}\dfrac{x^2-1}{|x-1|}=\lim\limits_{x\to 1^-}\dfrac{x^2-1}{-(x-1)}=-\lim\limits_{x\to 1^-}(x+1)=-2$

$\Rightarrow\quad\lim\limits_{x\to 1}\dfrac{x^2-1}{|x-1|}$ 不存在

4. $f(x)=\dfrac{21x^{-0.2}+8}{3x^{-0.2}+2}=\dfrac{\dfrac{21}{x^{0.2}}+8}{\dfrac{3}{x^{0.2}}+2}=\dfrac{21+8x^{0.2}}{3+2x^{0.2}}$, $A=\lim\limits_{x\to 0}f(x)=7,\quad B=\lim\limits_{x\to\infty}f(x)=4$

5. $\lim\limits_{(x,y)\to(1,2)}\dfrac{5x^2y}{x^2+y^2}=2$

6.

$4y^2+\ln x^2y=7$

微分 $\quad 8yy'+\dfrac{1}{x^2y}\left(2xy+x^2y'\right)=0$

整理 $\quad\left(8y+\dfrac{1}{y}\right)y'+\dfrac{2}{x}=0$

$\dfrac{8y^2+1}{y}y'=-\dfrac{2}{x}$

故 $\quad y'=\dfrac{dy}{dx}=-\dfrac{2y}{8xy^2+x}$

7.

$y=\dfrac{x(x-1)^{\frac{3}{2}}}{\sqrt{x+1}}$, $\quad x>1\quad$ 取對數 $\quad\ln y=\ln x+\dfrac{3}{2}\ln(x-1)-\dfrac{1}{2}\ln(x+1)$

微分 $\quad\dfrac{y'}{y}=\dfrac{1}{x}+\dfrac{3}{2(x-1)}-\dfrac{1}{2(x+1)}=\dfrac{2x^2-2+3x^2+3x-x^2+x}{2x(x-1)(x+1)}=\dfrac{2x^2+2x-1}{x(x-1)(x+1)}$

故 $\quad y'=\dfrac{2x^2+2x-1}{x(x-1)(x+1)}\cdot\dfrac{x(x-1)^{\frac{3}{2}}}{\sqrt{x+1}}=\dfrac{(2x^2+2x-1)\sqrt{x-1}}{\sqrt{(x+1)^3}}$.

8.

函數圖形上任一點 $(x, \sqrt{x})$

與 $(2, 0)$ 之距離 $d = \sqrt{(x-2)^2 + (\sqrt{x}-0)^2} = \sqrt{x^2 - 3x + 4}$

d 最小，即根號內函數 $x^2 - 3x + 4$ 要最小

求臨界點，微分=0 $2x - 3 = 0$

得 $x = \frac{3}{2}$

故最近點為 $\underline{\left(\frac{3}{2}, \frac{\sqrt{6}}{2}\right)}$

9.

$f(x) = (x-2)^3(x-1)$. Find the possible points of inflection $f'' = 0$.

$f' = 3(x-2)^2(x-1) + (x-2)^3 = (x-2)^2(4x-5)$

$f'' = 2(x-2)(4x-5) + (x-2)^2(4) = 6(x-2)(2x-3)$

由 $f'' = 0$ 得 $x = \frac{3}{2},\ 2$

將定義域分成三區間：$(-\infty, \frac{3}{2})$, $(\frac{3}{2}, 2)$, $(2, \infty)$

於三區間分別取測試數：$x = 0$, $x = 1.6$, $x = 3$ 得 $f'' > 0$, $f'' < 0$, $f'' > 0$

concave upward: $\underline{(-\infty, \frac{3}{2})}$, $\underline{(2, \infty)}$ concave downward: $\underline{(\frac{3}{2}, 2)}$

10.

$f(x) = x^2 e^{-x}$

Find the critical points of f. $f' = 0$

由 $f' = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x) = 0$ 得 $x = 0,\ 2$

There are critical points at $(0, 0)$, $(2, 4e^{-2})$

Second derivative test, $f'' = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} = e^{-x}(x^2 - 4x + 2)$

Check the critical points

$(0, 0)$ $f''(0) = (1)(0 - 0 + 2) = 2 > 0$

$(2, 4e^{-2})$ $f''(2) = (e^{-2})(4 - 8 + 2) = -2e^{-2} < 0$

relative maximum $\underline{(2, 4e^{-2})}$, relative minimum $\underline{(0, 0)}$

11.

Let $u = \ln x$, $du = \frac{1}{x}\,dx$.

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2\left(\frac{1}{x}\right)dx = \frac{(\ln x)^3}{3} + C$$

12.

$$\int \sec^2\frac{x}{2}\tan\frac{x}{2}\,dx = 2\int \sec\frac{x}{2}\left(\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right)dx$$

$$= \sec^2\frac{x}{2} + C \quad \text{or}$$

$$\int \sec^2\frac{x}{2}\tan\frac{x}{2}\,dx = 2\int \tan\frac{x}{2}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx = \tan^2\frac{x}{2} + C$$

13.

(a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\,d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

$$\begin{aligned}\int \sqrt{9-25x^2}\,dx &= \int (3\cos\theta)\frac{3}{5}\cos\theta\,d\theta\\ &= \frac{9}{5}\int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C\\ &= \frac{9}{10}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right) + C\end{aligned}$$

So, $\int_0^{3/5} \sqrt{9-25x^2}\,dx = \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\right]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}$.

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

So, $\int_0^{3/5} \sqrt{9-25x^2}\,dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}$.

14.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x + 1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$.

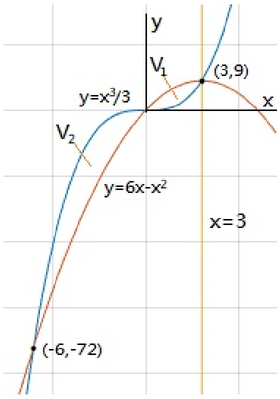
When $x = 1$, $2 = 2A + B + C$.

When $x = -1$, $0 = 2A + B - C$.

Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

$$\begin{aligned}\int_1^2 \frac{x+1}{x(x^2+1)}\,dx &= \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx\\ &= \left[\ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x\right]_1^2\\ &= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2\end{aligned}$$

15. $y = x^3/3$, $y = 6x - x^2$, $\Rightarrow$ $x^3/3 = 6x - x^2$ $\Rightarrow$ $x = 3, x = 0, x = -6$



$$V_1 = 2\pi\int_0^3 (3-x)(6x - x^2 - \frac{1}{3}x^3)dx = 2\pi\int_0^3 (x-3)(\frac{1}{3}x^3 + x^2 - 6x)dx$$

$$= 2\pi\int_0^3 (\frac{1}{3}x^4 - 9x^2 + 18x)dx = 2\pi\left[\frac{x^5}{15} - 3x^3 + 9x^2\right]_0^3 = \frac{162}{5}\pi$$

$$V_2 = 2\pi\int_{-6}^0 (3-x)(\frac{1}{3}x^3 + x^2 - 6x)dx = 2\pi\int_0^{-6} (x-3)(\frac{1}{3}x^3 + x^2 - 6x)dx$$

$$= 2\pi\int_0^{-6} (\frac{1}{3}x^4 - 9x^2 + 18x)dx = 2\pi\left[\frac{x^5}{15} - 3x^3 + 9x^2\right]_0^{-6} = \frac{4536}{5}\pi$$

$$V = V_1 + V_2 = \frac{162}{5}\pi + \frac{4536}{5}\pi = \frac{4698}{5}\pi$$

16. $\mathbf{F} = xyz\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z) = yz + x + 1$$

at $(2,1,1)$, $(\nabla\cdot\mathbf{F})_{(2,1,1)} = (1)(1) + 2 + 1 = 4$

17. $f(x,y) = x\tan y$, $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} = \tan y\,\mathbf{i} + x\sec^2 y\,\mathbf{j}$

at $(2, \frac{\pi}{4})$, $|\nabla f| = \left|\tan\frac{\pi}{4}\mathbf{i} + 2\sec^2\frac{\pi}{4}\mathbf{j}\right| = \sqrt{1+4^2} = \sqrt{17}$

18. $f(x,y,z) = 2x^2 + y^2 + 3z^2$, $g(x,y,z) = 2x - 3y - 4z - 49 = 0$

$\nabla f = 4x\,\mathbf{i} + 2y\,\mathbf{j} + 6z\,\mathbf{k}$, $\nabla g = 2\,\mathbf{i} - 3\,\mathbf{j} - 4\,\mathbf{k}$,

$\nabla f = \lambda\nabla g$ $\Rightarrow$ $4x = 2\lambda$, $2y = -3\lambda$, $6z = -4\lambda$,

$x = \frac{1}{2}\lambda$, $y = -\frac{3}{2}\lambda$, $z = -\frac{2}{3}\lambda$, $\Rightarrow$ $\lambda + \frac{9}{2}\lambda + \frac{8}{3}\lambda - 49 = 0$

$6\lambda + 27\lambda + 16\lambda - (49)(6) = 0$ $\Rightarrow$ $\lambda = 6$ $\Rightarrow$ $x = 3, y = -9, z = -4$

$$f_{\max} = 2(3)^2 + (-9)^2 + 3(-4)^2 = 18 + 81 + 48 = 147$$

19. $\iint_R (x^2 + y)dA$ $\quad$ $x = r\cos\theta$, $y = r\sin\theta$, $R: 1 \le x^2 + y^2 = r^2 \le 5$

$$= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^2\cos^2\theta + r\sin\theta)r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{5}} (r^3\cos^2\theta + r^2\sin\theta)dr\,d\theta$$

$$= \int_0^{2\pi}[\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta]_1^{\sqrt{5}}d\theta = 6\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta + (\frac{5\sqrt{5}-1}{3})\int_0^{2\pi}\sin\theta\,d\theta$$

$$= 6\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} - \frac{5\sqrt{5}-1}{3}[\cos\theta]_0^{2\pi} = 6\pi$$

20. $f(x,y,z)=z^2-2x^2-2y^2-12=0$, $\nabla f\big|_{(1,-1,4)}=(-4x\mathbf{i}-4y\mathbf{j}+2z\mathbf{k})_{(1,-1,4)}=-4\mathbf{i}+4\mathbf{j}+8\mathbf{k}$

The tangent plane is therefore the plane

$$(x-1)-(y+1)-2(z-4)=0, \quad or \quad x-y-2z+6=0$$

21. $x=s^2-t^2, \quad y=\dfrac{s}{t}, \quad w=2xy$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}=2y(-2t)+2x(-\frac{s}{t^2})=-4(\frac{s}{t})t-2(s^2-t^2)\frac{s}{t^2}$$

$$=-4s-2(s^2-t^2)\frac{s}{t^2}=\frac{-4st^2-2s^3+2st^2}{t^2}=\frac{-2st^2-2s^3}{t^2}$$

22. $F(x,y,z)=3x^2z-x^2y^2+2z^3+3yz-5=0$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{-2x^2y+3z}{3x^2+6z^2+3y}=\frac{2x^2y-3z}{3x^2+6z^2+3y}$$

23. $z=4-x^2-2y^2, \quad R: x^2+2y^2\le 4, \quad V=\iint_R z\,dA=\iint_R (4-x^2-2y^2)dA$

Let $x=u, \quad y=\dfrac{v}{\sqrt{2}}, \qquad x^2+2y^2=u^2+v^2\le 4 \qquad S: \quad u^2+v^2\le 4$

coordinate transformation from (x,y) to (u,v), $\quad x=g(u,v), \quad y=h(u,v)$

$$\iint_R f(x,y)dA=\iint_R f(x,y)dxdy=\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & \dfrac{1}{\sqrt{2}}\end{vmatrix}=\frac{1}{\sqrt{2}}$$

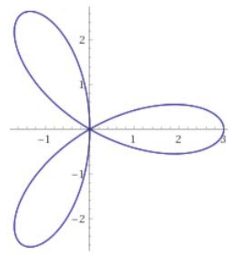
$$V=\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv=\iint_S (4-u^2-v^2)\frac{1}{\sqrt{2}}dudv=\frac{1}{\sqrt{2}}\iint_S (4-u^2-v^2)dudv$$

Taking coordinate transformation again, let $u=r\cos\theta, \quad v=r\sin\theta, \quad u^2+v^2=r^2$

$$V=\frac{1}{\sqrt{2}}\int_0^{2\pi}\int_0^2 (4-r^2)\,rdrd\theta=\frac{1}{\sqrt{2}}\int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta=\frac{1}{\sqrt{2}}(8-4)(2\pi)=4\sqrt{2}\pi$$

24. $r=3\cos 3\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	3	0	-3	0	3	0	-3



$$\mathrm{A}=\frac{1}{2}\int r^2 d\theta=\frac{1}{2}\int_0^{\pi} 9\cos^2 3\theta\, d\theta=\frac{9}{2}\int_0^{\pi}\frac{1+\cos 6\theta}{2}d\theta=\frac{9}{2}\left[\frac{1}{2}\theta+\frac{\sin 6\theta}{12}\right]_0^{\pi}=\frac{9\pi}{4}$$