國立宜蘭大學 108 年度微積分競試 解答

1. 收斂	2. $\frac{3}{4}$
3. $r=\frac{9\cos\theta}{\sin^2\theta}$	4. 4
5. -3	6. $\frac{1}{2}$
7. $\frac{1}{2}$	8. 10!
9. $\frac{1}{2}$	10. $(-1)^{n-1}(n-1)!(x+1)^{-n}$
11. $-\frac{1}{130}$	12. $51e^4+21$
13. $\frac{\pi}{4}$	14. 0
15. $\frac{2^{\sin\theta}}{\ln 2}+C$	16. $\frac{\pi^2}{72}$
17. $\frac{124}{5}$	18. $\frac{\partial z}{\partial x}=\frac{1}{y(x+z)-1};\frac{\partial z}{\partial y}=-\frac{z(x+z)}{y(x+z)-1}$
19. $\frac{7\sqrt{5}}{10}$	20. 最大變化率為 1，方向為 $\mathbf{j}$ $<0,1>$
21. 48 為最小值	22. 2

1. $\sum_{n=1}^{\infty} 2(-\frac{1}{3})^n$ 為等比級數，公比為$-\frac{1}{3}$ $\therefore$ 收斂

2. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2}) = \frac{1}{2}[(1-\frac{1}{3}) + (\frac{1}{2}-\frac{1}{4}) + (\frac{1}{3}-\frac{1}{5}) + (\frac{1}{4}-\frac{1}{6}) + (\frac{1}{5}-\frac{1}{7}) + \cdots$

$= \frac{1}{2}(1+\frac{1}{2}) = \frac{3}{4}$

3. $y^2 = 9x$，令 $x = r\cos\theta,\quad y = r\sin\theta \;\Rightarrow\; r^2\sin^2\theta = 9r\cos\theta \;\Rightarrow\; r = \frac{9\cos\theta}{\sin^2\theta}$

4. 令 $y = f(x) = 3 + \cfrac{x}{3 + \cfrac{x}{3 + \cfrac{x}{3 + \cfrac{x}{\cdots}}}} \;\Rightarrow\; y = 3 + \frac{x}{y} \;\Rightarrow\; y^2 = 3y + x \;\Rightarrow\; y^2 - 3y - x = 0$

$y = \frac{3 \pm \sqrt{9+4x}}{2} \quad \because\; f(4) > 0 \;\Rightarrow\; f(4) = \frac{3+\sqrt{9+16}}{2} = 4$

5. $f(x) = \frac{(x+1)(x-2)(x-3)}{(x-1)(x+2)(x+3)}$

$f'(-1) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{1}{h}\cdot[\frac{h(h-3)(h-4)}{(h-2)(h+1)(h+2)} - 0]$

$= \lim_{h\to 0}\frac{(h-3)(h-4)}{(h-2)(h+1)(h+2)} = \frac{(-3)(-4)}{(-2)(1)(2)} = -3$

6. $k = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}$

7. $x^2 + 4y^2 = 4 \;\Rightarrow\; 2x + 8y\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{x}{4y}$

$x = \sqrt{2},\quad y = -\frac{1}{\sqrt{2}} \;\Rightarrow\; \left.\frac{dy}{dx}\right|_{(\sqrt{2},-\frac{1}{\sqrt{2}})} = -\frac{\sqrt{2}}{4(-\frac{1}{\sqrt{2}})} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$

8. $f(x) = x(x+1)(x+2)(x+3)\cdots(x+9)(x+10)$

$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{1}{h}\cdot[h(h+1)(h+2)\cdots(h+10) - 0] = (1)(2)(3)\cdots(9)(10) = 10!$

9. $\lim_{x\to -\infty}(x\sqrt{x^2-1}+x^2) = \lim_{x\to -\infty}(x\sqrt{x^2-1}+x^2)\frac{x\sqrt{x^2-1}-x^2}{x\sqrt{x^2-1}-x^2} = \lim_{x\to -\infty}\frac{x^2(x^2-1)-x^4}{x\sqrt{x^2-1}-x^2}$

$= \lim_{x\to -\infty}\frac{-x^2}{x\sqrt{x^2-1}-x^2}$ 令 $z = -x;\quad x\to -\infty, z\to\infty$

$= \lim_{z\to\infty}\frac{-z^2}{-z\sqrt{z^2-1}-z^2} = \lim_{z\to\infty}\frac{z^2}{z\sqrt{z^2-1}+z^2} = \lim_{z\to\infty}\frac{z^2/z^2}{(z\sqrt{z^2-1}+z^2)/z^2} = \lim_{z\to\infty}\frac{1}{\sqrt{1-1/z^2}+1} = \frac{1}{2}$

10. $y=\ln(x+1)$, $\dfrac{dy}{dx}=(x+1)^{-1}$, $\dfrac{d^2y}{dx^2}=(-1)(x+1)^{-2}$, $\dfrac{d^3y}{dx^3}=(-1)(-2)(x+1)^{-3}=2(x+1)^{-3}$

$\dfrac{d^4y}{dx^4}=2(-3)(x+1)^{-4}$, $\dfrac{d^5y}{dx^5}=2(-3)(-4)(x+1)^{-5}$, $\cdots$, $\dfrac{d^ny}{dx^n}=(-1)^{n-1}(n-1)!(x+1)^{-n}$

11. $Z=\ln(x^2+y^2)$, $\dfrac{\partial Z}{\partial x}=\dfrac{2x}{x^2+y^2}$, $\dfrac{\partial Z}{\partial y}=\dfrac{2y}{x^2+y^2}$, $x=2,\quad y=3,\quad dx=0.02,\quad dy=-0.03$

$$dZ=\frac{\partial Z}{\partial x}dx+\frac{\partial Z}{\partial y}dy=\frac{4}{13}(0.02)+\frac{6}{13}(-0.03)=\frac{0.08}{13}-\frac{0.18}{13}=-\frac{0.1}{13}=-\frac{1}{130}$$

12. $f(x,y)=(7e^{x+2y}+4)(e^{x^2}+y^2+2)$, $\dfrac{\partial f}{\partial x}=(7e^{x+2y})(e^{x^2}+y^2+2)+(7e^{x+2y}+4)(e^{x^2})2x$

$$\frac{\partial f}{\partial x}(2,-1)=7e^0(e^4+3)+(7e^0+4)e^4(4)=7e^4+21+44e^4=51e^4+21$$

13. $\int_0^1\sqrt{1-x^2}dx$ 令 $x=\sin\theta\Rightarrow\sqrt{1-x^2}=\cos\theta, dx=\cos\theta d\theta$, $x=0,\theta=0; x=1,\theta=\dfrac{\pi}{2}$

$$=\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\int_0^{\frac{\pi}{2}}(\frac{1+\cos2\theta}{2})d\theta=\left[\frac{1}{2}\theta+\frac{\sin2\theta}{4}\right]_0^{\frac{\pi}{2}}=(\frac{\pi}{4}+\frac{\sin\pi}{4})-(0+\frac{\sin0}{4})=\frac{\pi}{4}$$

14. $f(x)=x^2$ 及 $f(x)=8-\cos x$ 為偶函數，$f(x)=\tan x$ 為奇函數

$\therefore\ f(x)=\dfrac{x^2\tan x}{8-\cos x}$ 為奇函數 $\Rightarrow\ \int_{-1}^{1}\dfrac{x^2\tan x}{8-\cos x}dx=0$

15. $\int 2^{\sin\theta}\cos\theta\,d\theta$ 令 $z=\sin\theta,\quad dz=\cos\theta\,d\theta$

$$\int 2^z dz=\frac{2^z}{\ln2}+C=\frac{2^{\sin\theta}}{\ln2}+C$$

16. $\int_0^{\frac{1}{2}}\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}dx$ 令 $z=\sin^{-1}x,\quad dz=\dfrac{dx}{\sqrt{1-x^2}};\quad x=0,z=0;\quad x=\dfrac{1}{2},z=\dfrac{\pi}{6}$

$$=\int_0^{\frac{\pi}{6}}zdz=\left[\frac{1}{2}z^2\right]_0^{\frac{\pi}{6}}=\frac{1}{2}(\frac{\pi}{6})^2=\frac{\pi^2}{72}$$

17. $y=\int_1^x\sqrt{\sqrt{t}-1}\,dt$, $\dfrac{dy}{dx}=\sqrt{\sqrt{x}-1}$, $ds=\sqrt{1+(\dfrac{dy}{dx})^2}\,dx=\sqrt{1+\sqrt{x}-1}\,dx=x^{1/4}dx$

$$L=\int ds=\int_1^{16}x^{1/4}dx=\frac{4}{5}[x^{5/4}]_1^{16}=\frac{4}{5}(32-1)=\frac{124}{5}$$

18. $F(x,y,z)=yz-\ln(x+z)=0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{-\frac{1}{x+z}(1)}{y-\frac{1}{x+z}(1)}=\frac{1}{y(x+z)-1}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{z}{y-\frac{1}{x+z}}=-\frac{z(x+z)}{y(x+z)-1}$$

19 取梯度 $\nabla g(s,t)=\sqrt{t}\mathbf{i}+\dfrac{s}{2\sqrt{t}}\mathbf{j}$ 在點(2, 4)之梯度值 $\nabla g(2,4)=2\mathbf{i}+\dfrac{1}{2}\mathbf{j}$

單位向量 $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{1}{\sqrt{5}}(2\mathbf{i}-\mathbf{j})$

故 $D_{\mathbf{u}}g(2,4)=\nabla g(2,4)\cdot\mathbf{u}=\dfrac{1}{\sqrt{5}}(4-\dfrac{1}{2})=\dfrac{7}{2\sqrt{5}}=\dfrac{7\sqrt{5}}{10}$

20 取梯度 $\nabla f(x,y)=y\cos(xy)\mathbf{i}+x\cos(xy)\mathbf{j}$

在點(1, 0)之梯度值 $\nabla f(1,0)=0\mathbf{i}+1\mathbf{j}=<0,1>$ 即為 f 最大變化率之方向

最大變化率 $\|\nabla f(1,0)\|=\sqrt{0+1}=1$

21 $f(x,y,z)=x^2+y^2+z^2;\quad x+y+z=12$

Constraint $g(x,y,z)=x+y+z=12$, and $\nabla f=\lambda\nabla g$

$\nabla f=<2x,\ 2y,\ 2z>,\quad \lambda\nabla g=<\lambda,\ \lambda,\ \lambda>$

then $x=y=z=\dfrac{\lambda}{2}$ 代入 $g=\dfrac{3\lambda}{2}=12$ 得 $\lambda=8$

故 $f(4,4,4)=16+16+16=48$ 為 minimum

22.

$$\begin{aligned}\iint_R x\sec^2 y\,dA &= \int_0^2 \int_0^{\pi/4} x\sec^2 y\,dy\,dx\\ &= \int_0^2 x\tan y\Big|_0^{\pi/4}dx\\ &= \int_0^2 x(1-0)dx\\ &= \frac{1}{2}x^2\Big|_0^2\\ &= \frac{1}{2}(4-0)=2\end{aligned}$$