國立宜蘭大學 109 年度微積分競試 解答

1. 收斂	2. 發散
3. $\sum_{n=0}^{\infty}(n+1)x^n$	4. $\left\langle \cos\frac{3y}{z}, -\frac{3x}{z}\sin\frac{3y}{z}, \frac{3xy}{z^2}\sin\frac{3y}{z} \right\rangle$
5. $\frac{5}{2}$	6. $-\infty$
7. 0	8. 0
9. $y=6x-21$ and $y=6x-9$	10. 5.00400
11. $\frac{2xy^4\sec^2 x^2y^4-3}{2y\left(1-2x^2y^2\sec^2 x^2y^4\right)}$	12. 5
13. $\frac{13}{2}$	14. $\sqrt{3}\sec\theta-3\ln\lvert\sec\theta+\tan\theta\rvert+C$
15. $\frac{1}{4}\left[2\left(t^2-1\right)\ln\lvert t+1\rvert-t^2+2t\right]+C$	16. $\frac{1}{\pi}\left(e^{\sin\frac{\pi^2}{2}}-1\right)$
17. $\ln(\sqrt{2}+1)$	18. $\frac{\partial z}{\partial x}=\frac{1-y\cos(xy+z)}{\cos(xy+z)}$
19. max $\frac{\pi-2}{3}$ min $\frac{2-\pi}{3}$	20. $-\frac{1}{2}\left(\cos(16)-1\right)$
21. $\frac{\pi}{48}\left(e^6-1\right)$	22. $2a+b=25$

1.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3-1}{n^4+1} = \sum_{n=1}^{\infty} (-1)^n \cdot b_n,$$

① $b_n = \frac{n^3-1}{n^4+1} > 0$, for $n \geq 2$

② $\lim_{n\to\infty} b_n = 0$

$\because$ By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3-1}{n^4+1}$ converges.

2.

Use the Limit Comparison Test with $a_n = \frac{n^7-1}{n^8+1}$, and $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^7-1}{n^8+1} \cdot n = \lim_{n\to\infty} \frac{n^8-n}{n^8+1} = 1 > 0$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergence series, the series $\sum_{n=1}^{\infty} \frac{n^7-1}{n^8+1}$ also divergence

3.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (n+1) \cdot x^n$$

4.

$$f(x,y,z) = x \cdot \cos\left(\frac{3y}{z}\right)$$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$= \cos\left(\frac{3y}{z}\right) \vec{i} + \left(\frac{-3x}{z}\right) \cdot \sin\left(\frac{3y}{z}\right) \vec{j} + \frac{3xy}{z^2} \cdot \sin\left(\frac{3y}{z}\right) \vec{k}$$

5.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (cx^2-3) = 4c-3; \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (cx+2) = 2c+2$$

$\because f$ is continuous $\Rightarrow \lim_{x\to 2} f(x)$ exists $\Rightarrow 4c-3 = 2c+2$, $c = \frac{5}{2}$

6.

$$\lim_{x \to 3^-} \frac{x^2+x+2}{x^2-2x-3} = \lim_{x \to 3^-} \frac{x^2+x+2}{(x-3)(x+1)} = -\infty$$

7.

$$\lim_{x \to \infty} \frac{3x^2}{x^3-x^2+x} = \lim_{x \to \infty} \frac{3}{x-1+\frac{1}{x}} = 0$$

8.

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} |x| = -\lim_{x \to 0^-} x = 0$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} |x| = +\lim_{x \to 0^+} x = 0 \qquad \therefore \lim_{x \to 0} g(x) = 0$$

9.

$$y + \frac{3}{2x-5} = 0$$

$$\Rightarrow \quad y = -3(2x-5)^{-1}$$

$$y' = \frac{dy}{dx} = (-3)(-1)(2x-5)^{-2}(2x-5)' = \frac{6}{(2x-5)^2}$$

Set the equations of tangent lines (to the curve) as

$$y = m(x-x_0) + y_0$$

$$\Rightarrow m = y'(x_0) = 6$$

i.e. $\frac{6}{(2x_0-5)^2} = 6$

$\Rightarrow x_0 = 3$ or $x_0 = 2$

$\Rightarrow y_0 = -3(2x_0-5)^{-1} = -3$; Similarly, $y_0 = 3$

The tangent lines pass through $(3,-3)$ and $(2,3)$, respectively, and therefore, their equations yield:

$$\mathbf{y = 6x-21} \quad \text{and} \quad \mathbf{y = 6x-9}$$

10.

Approximate $\sqrt[3]{125.3}$ to five decimal place

Let $f(x) = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$\because f'(x) = \frac{df(x)}{dx}$

$df(x) = f'(x)\,dx$

$\Delta f(x) \cong f'(x)\,\Delta x$

$\Delta f(x) = f(x+\Delta x) - f(x) \cong f'(x)\,\Delta x$

$\therefore \; f(x+\Delta x) \cong f(x) + f'(x)\,\Delta x$

Setting $x = 125$, and $\Delta x = 0.3$ yields

$$\sqrt[3]{125.3} \cong \sqrt[3]{125} + \frac{1}{3}(125)^{-\frac{2}{3}} \cdot (0.3)$$

$$= 5 + \frac{1}{3} \cdot \frac{1}{25} \cdot (0.3)$$

$$= 5 + \frac{0.1}{25}$$

$$= 5.00400$$

$\therefore \sqrt[3]{125.3} \cong$ **5.00400**

11.

$$\tan(x^2 y^4) = 3x + y^2$$

Differentiating the equation with respect to x yields

$$[\sec^2(x^2y^4)]\frac{d(x^2y^4)}{dx} = 3\frac{d(x)}{dx} + 2y\left(\frac{dy}{dx}\right)$$

$$\left[2xy^4 + 4x^2y^3\left(\frac{dy}{dx}\right)\right]\sec^2(x^2y^4) = 3 + 2y\left(\frac{dy}{dx}\right)$$

$$2y\left(\frac{dy}{dx}\right)\left[1 - 2x^2y^2\sec^2(x^2y^4)\right] = 2xy^4\sec^2(x^2y^4) - 3$$

$$y' = \frac{dy}{dx} = \frac{2xy^4\sec^2(x^2y^4) - 3}{2y\left[1 - 2x^2y^2\sec^2(x^2y^4)\right]}$$

12.

According to the Mean-Value Theorem,

$$\frac{f(-1)-f(-5)}{(-1)-(-5)} = f'(m), \quad \text{where } m \in (-5,-1)$$

It is required that $f'(m) \le 2$, and that

$f(-5) = -3$.

Then we have

$$\frac{f(-1)-(-3)}{(-1)-(-5)} \le 2$$

$\therefore \quad f(-1) \le 5$

The largest possible value of $f(-1)$ is **5**.

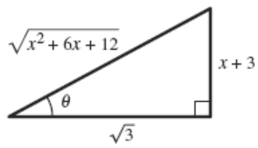
13. $\int_1^4 (3-|x-3|)dx = \int_1^3 x\,dx + \int_3^4 (6-x)dx = \frac{1}{2}x^2 \Big|_1^3 + \left(6x - \frac{1}{2}x^2\right)\Big|_3^4 = 4 + 16 - \frac{27}{2} = \frac{13}{2}$

14.

$x^2 + 6x + 12 = x^2 + 6x + 9 + 3 = (x+3)^2 + \left(\sqrt{3}\right)^2$

Let $x + 3 = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\, d\theta$.

$\sqrt{x^2 + 6x + 12} = \sqrt{(x+3)^2 + \left(\sqrt{3}\right)^2} = \sqrt{3}\sec\theta$



$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 + 6x + 12}}\,dx &= \int \frac{\sqrt{3}\tan\theta - 3}{\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\, d\theta \\
&= \int \sqrt{3}\sec\theta\tan\theta\, d\theta - 3\int \sec\theta\, d\theta \\
&= \sqrt{3}\sec\theta - 3\ln\left|\sec\theta + \tan\theta\right| + C \\
&= \sqrt{3}\left(\frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}}\right) - 3\ln\left|\frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}} + \frac{x+3}{\sqrt{3}}\right| + C \\
&= \sqrt{x^2 + 6x + 12} - 3\ln\left|\sqrt{x^2 + 6x + 12} + (x+3)\right| + C
\end{aligned}$$

15.

$$dv = t\,dt \quad \Rightarrow \quad v = \int t\,dt = \frac{t^2}{2}$$

$$u = \ln(t+1) \Rightarrow du = \frac{1}{t+1}\,dt$$

$$\begin{aligned}\int t\ln(t+1)\,dt &= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int \frac{t^2}{t+1}\,dt \\ &= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int\left(t-1+\frac{1}{t+1}\right)dt \\ &= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right] + C \\ &= \frac{1}{4}\left[2(t^2-1)\ln|t+1| - t^2 + 2t\right] + C\end{aligned}$$

16. $\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x dx$ $\qquad u = \sin \pi x,\ du = \pi \cos \pi x$

$$= \int \frac{1}{\pi} e^u du = \frac{1}{\pi} e^{\sin \pi x}\Big|_0^{\frac{\pi}{2}} = \frac{1}{\pi}\left(e^{\sin\frac{\pi^2}{2}} - 1\right)$$

17.

$$y = \ln(\sec x) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sec x\ \tan x}{\sec x} = \tan x \quad \Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x, \text{ so}$$

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\ dx = \int_0^{\pi/4} \sec x\,dx = \Big[\ln(\sec x + \tan x)\Big]_0^{\pi/4}$$

$$= \ln(\sqrt{2}+1) - \ln(1+0) = \ln(\sqrt{2}+1)$$

18. $F = x + y^y - sin(xy+z) \rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1 - y\,cos(xy+z)}{cos(xy+z)}$

19.

$$f(x) = 2x - \frac{4}{3}\arctan 3x$$

$$f'(x) = 2 - \frac{4}{3}\cdot\frac{(3x)'}{1+(3x)^2}$$

$$= 2\left(1 - \frac{2}{1+9x^2}\right)$$

$$= 2\,\frac{(9x^2-1)}{1+9x^2}$$

$$= \frac{2\,(3x+1)(3x-1)}{1+9x^2}$$

To find the critical numbers,

Setting $f'(c)=0$ or that $f'(c)$ does not exist, (however, no number satisfies this condition)

yields

$$c=-\frac{1}{3} \quad \text{or} \quad c=\frac{1}{3}$$

Using first-derivative test for determining whether $f(c)$ an extreme value or not.

critical number		$-\frac{1}{3}$		$\frac{1}{3}$	
number for trial	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$ increasing or decreasing	$\nearrow$		$\searrow$		$\nearrow$

$\therefore\ f(-\frac{1}{3}) =$ relative **max**imum

$$= 2(-\tfrac{1}{3}) - \tfrac{4}{3}\arctan[3(-\tfrac{1}{3})] = \tfrac{-2}{3} - \tfrac{4}{3}(-\tfrac{\pi}{4})$$

Similarly, $f(\frac{1}{3}) =$ relative **mini**mum $= \frac{2-\pi}{3}$ $= \frac{\pi-2}{3}$

20. $\int_0^4 \int_0^{x^2} x\cos(y)dydx = \int_0^4 x\sin(y)|_0^{x^2} dx = \int_0^4 x\sin(x^2)dx = \frac{1}{2}\int_0^{16} \sin(u)du = -\frac{1}{2}(\cos(16)-1) =$

0.979

21. $R: 0 \le y \le \frac{1}{\sqrt{2}}, y \le x \le \sqrt{1-y^2} \rightarrow 0 \le \theta \le \frac{\pi}{4}, 0 \le r \le 1$

$$\int_0^{\frac{1}{\sqrt{2}}}\int_y^{\sqrt{1-y^2}} e^{6x^2+6y^2}dxdy = \int_0^{\frac{\pi}{4}}\int_0^1 e^{6r^2} rdrd\theta = \frac{1}{12}\int_0^{\frac{\pi}{4}}\int_0^6 e^u dud\theta = \frac{\pi}{48}(e^6-1) = 26.34$$

22. $\frac{1}{4\times 2}\int_0^4\int_0^2 (ax+by)dydx = \frac{1}{8}\int_0^4 axy + \frac{1}{2}by^2\Big|_0^2 dx = \frac{1}{8}\int_0^4 2ax + 2b\,dx = 2a + b = 25$