國立宜蘭大學 110 年度微積分競試 解答

1. 收斂	2. 發散
3. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$	4. $\langle 1,\ 12,\ 0 \rangle$
5. 1	6. -3
7. e^{-1}	8. 1
9. $-2\theta \sin(\theta^2)$	10. $\frac{8(1-\cot x - x\csc^2 x)}{(1-\cot x)^2}$
11. $\frac{x-4y}{4x-y}$	12. $\sqrt{17}$
13. $\ln\left\lvert\frac{x+2}{x+3}\right\rvert + C$	14. $\frac{1}{2\ln 2}\ln\lvert 2^{2x} - 2^{-2x}\rvert + C$
15. $\frac{1}{\ln ab}(ab)^x + C$	16. $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$
17. 147	18. $2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$
19. -86.11	20. $\langle -(e^2-1)\cos(6), (e^2-2)\sin(6) \rangle$
21. $e^4 - 1$	22. $-\frac{1}{3}\ or\ \frac{1}{2}$

1.

$$\sum_{n=0}^{\infty} \frac{\sin(n+\frac{1}{2})\pi}{1+\sqrt{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}, \quad a_n = \frac{1}{1+\sqrt{n}} > 0 \text{ for } n \geq 0$$

$$a_{n+1} = \frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}} = a_n \text{ . \& } \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0 \text{ .}$$

$\therefore$ con. alternating series test

2.

$b_n = \frac{1}{n}$. $\Sigma \frac{1}{n}$. $p=1$ div. 4故 limit comparison

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{ne^n+n}{ne^n+1} = \lim_{n\to\infty} \frac{1+\frac{1}{e^n}}{1+\frac{1}{ne^n}} = 1 \in R$$

$\therefore$ div.

3.

$$\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \Rightarrow$$

$$f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$= 1 - \tfrac{1}{2}x^4 + \tfrac{1}{24}x^8 - \tfrac{1}{720}x^{12} + \cdots$$

4.

$f(x, y, z) = xe^{2yz}$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \left\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\right\rangle$

(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0\rangle$

5.

that $f(x) = \dfrac{3}{2x+f(x)}$

Assumed that the denominator "$2x+f(x)$" is not zero, then we have

$$[f(x)]^2 + (2x)f(x) - 3 = 0$$

$\Rightarrow$ $f(x) = -x + \sqrt{x^2+3}$

or $f(x) = -x - \sqrt{x^2+3}$

however,

$$f(1) = \cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{\vdots}}}$$

$\Rightarrow$ $f(1)$ is positive.

$\Rightarrow$ $f(1) = 1$

6.

$$\lim_{x\to\infty} \sqrt{x(x-3)} - \sqrt{x(x+3)}$$

$$= \lim_{x\to\infty}\left[\sqrt{x(x-3)} - \sqrt{x(x+3)}\right]\cdot\left[\frac{\sqrt{x(x-3)}+\sqrt{x(x+3)}}{\sqrt{x(x-3)}+\sqrt{x(x+3)}}\right]$$

$$= \lim_{x\to\infty} \frac{x(x-3)-x(x+3)}{\sqrt{x(x-3)}+\sqrt{x(x+3)}}$$

$$= -6 \lim_{x\to\infty} \frac{x}{\sqrt{x(x-3)}+\sqrt{x(x+3)}}$$

$$= -6 \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{\sqrt{x(x-3)}+\sqrt{x(x+3)}}{x}}$$

$$= -6 \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{3}{x}}+\sqrt{1+\frac{3}{x}}}$$

$$= -6 \lim_{x\to\infty} \frac{1}{1+1}$$

$$= -3$$

7.

$$\lim_{x\to1^+} x^{\frac{1}{1-x}}$$

Let $y = \lim_{x\to1^+} x^{\frac{1}{1-x}}$

$$\ln y = \ln \lim_{x\to1^+} x^{\frac{1}{1-x}}$$

$$= \lim_{x\to1^+} \ln x^{\frac{1}{1-x}}$$

$$\ln x^{\frac{1}{1-x}} = \frac{1}{1-x}\ln x = \frac{\ln x}{1-x}$$

$$\lim_{x\to1^+} \frac{\ln x}{1-x} \Rightarrow \frac{0}{0} \text{ type.}$$

L'Hôpital's rule can be applied here.

$$\Rightarrow \lim_{x\to1^+} \frac{\frac{1}{x}}{-1}$$

$$= -1$$

$$\Rightarrow \ln y = -1$$

$$y = e^{-1}$$

$$\therefore \lim_{x\to1^+} x^{\frac{1}{1-x}} = e^{-1}$$

8.

$$\lim_{x\to0^+} x\tan\left(\frac{\pi}{2}-x\right).$$

$$= \lim_{x\to0^+} x\left[\frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)}\right]$$

$$\Rightarrow \quad 0\cdot\frac{1}{0} \Rightarrow \frac{0}{0} \text{ type}$$

L'Hôpital's rule can be applied.

$$\Rightarrow \lim_{x\to0^+} \frac{\sin(\frac{\pi}{2}-x) - x\left[\cos(\frac{\pi}{2}-x)\right](-1)}{\left[-\sin(\frac{\pi}{2}-x)\right](-1)}$$

$$= \frac{1-0\cdot0\cdot(-1)}{1} = 1.$$

$$\therefore \lim_{x\to0^+} x\tan\left(\frac{\pi}{2}-x\right)$$

$$= 1$$

9.

$$\frac{d\,f(\theta)}{d\theta} = \frac{d\cos(\theta^2)}{d\theta} = (-)\sin(\theta^2)\cdot 2\theta$$

10.

$$y' = \frac{d}{dx}\left(\frac{8x}{1-\cot(x)}\right)$$

$$= \frac{8\cdot(1-\cot(x)) - 8x\cdot(-)(-)\csc^2(x)}{[1-\cot(x)]^2}$$

$$= \frac{8(1-\cot(x) - x\cdot\csc^2(x))}{[1-\cot(x)]^2}$$

11.

$$\frac{d}{dx}(x^2 - 8xy + y^2) = 8$$

$$2x - 8y - 8x\cdot y' + 2y\cdot y' = 0$$

$$y' = \frac{2x-8y}{8x-2y} = \frac{x-4y}{4x-y}$$

12.

$$\begin{aligned} f_x(x,y) &= \tan y \\ f_y(x,y) &= x\sec^2 y \end{aligned}$$

$$\begin{aligned} \nabla f(x,y) &= f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} \\ &= \tan y\mathbf{i} + x\sec^2 y\mathbf{j} \\ \nabla f\left(2,\frac{\pi}{4}\right) &= \tan\frac{\pi}{4}i + 2\sec^2\frac{\pi}{4}j \\ &= i + 4j \end{aligned}$$

The maximum value of directional derivative is equal to $\left\|\nabla f(2,\frac{\pi}{4})\right\| = \sqrt{1+4^2} = \sqrt{17}$

13.

4.

$$\int \frac{1}{x^2+5x+6}dx$$

$$= \int \frac{1}{(x+2)(x+3)}dx$$

$$= \int\left(\frac{1}{x+2} + \frac{-1}{x+3}\right)dx$$

$$= \ln|x+2| - \ln|x+3| + C$$

$$= \ln\left|\frac{x+2}{x+3}\right| + C.$$

14.

2. $\int \frac{2^{2x}+2^{-2x}}{2^{2x}-2^{-2x}}dx$

$u = 2^{2x}-2^{-2x}$

$du = 2(2^{2x}+2^{-2x})\ln 2$

原式 $= \int \frac{\frac{1}{2\ln 2}}{u}du$

$= \frac{1}{2\ln 2}\cdot \ln|2^{2x}-2^{-2x}|+C$

15.

3. $\int a^x b^x dx$

$= \int (ab)^x dx$

$= \frac{1}{\ln ab}(ab)^x + C$

16.

6. $F'(x) = x^2 \ln x$

$F(x) = \int F'(x)dx$

$= \int x^2 \ln x \, dx$

$= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

17.

令 $g(x,y,z) = 2x - 3y - 4z$

$$\begin{aligned} f(x,y,z) &= 2x^2+y^2+3z^2 \\ \nabla f(x,y,z) &= 4x\mathbf{i}+2y\mathbf{j}+6z\mathbf{k} \\ \lambda\nabla_g &= 2\lambda\mathbf{i}-3\lambda\mathbf{j}-4\lambda\mathbf{k} \end{aligned}$$

$$\begin{aligned} 4x &= 2\lambda \\ 2y &= -3\lambda \\ 6z &= -4\lambda \\ 2x-3y-4z &= 49 \end{aligned}$$

解得 $x = 3,\ y = -9,\ z = -4,\ \lambda = 6$

$$\begin{aligned} f(3,-9,-4) &= 2(3)^2+(-9)^2+3(-4)^2 \\ &= 147 \end{aligned}$$

18.

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= 2xy\left(\cos t\right) + \left(x^2 - 2y\right)e^t \\
&= 2\left(\sin t\right)\left(e^t\right)\left(\cos t\right) + \left(\sin^2 t - 2e^t\right)e^t \\
&= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}
\end{aligned}$$

19.

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n \approx 1 - x^3 + x^6 - x^9 \rightarrow$$

$$\int_0^2 \frac{1}{1+x^3}\,dx \approx \int_0^2 1 - x^3 + x^6 - x^9\,dx = -86.11$$

20.

$$\nabla f = \langle (e^x - 1)\cos(y), -(e^x - x)\sin(y) \rangle \rightarrow$$
$$-\nabla f(2,6) = \langle -(e^2 - 1)\cos(6), (e^2 - 2)\sin(6) \rangle$$

21.

$$\begin{aligned}
\int_0^1 \int_{2x}^2 4e^{y^2}\,dy\,dx &= 4\int_0^2 \int_0^{\frac{1}{2}y} e^{y^2}\,dx\,dy \\
&= 4\int_0^2 \left[xe^{y^2}\right]_0^{\frac{1}{2}y} dy \\
&= 4\int_0^2 \left(\frac{1}{2}ye^{y^2}\right) dy \\
&= \int_0^2 \left(2ye^{y^2}\right) dy \\
&= \left[e^{y^2}\right]_0^2 \\
&= e^4 - 1
\end{aligned}$$

22.

$$div\vec{F} = 6a^2 - a - 1 = 0 \rightarrow a = -\frac{1}{3} \ or \ \frac{1}{2}$$