國立宜蘭大學 111 年度微積分競試 解答

1. $x \le \frac{2}{3} \cup x > 1$	2. 不存在
3. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$	4. $-1,\ 1.477$
5. 6.0028	6. 8
7. $\pi^x x^{\pi^x}\left[\frac{1}{x} + (\ln\pi)(\ln x)\right]$	8. $\frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y$
9. min. $-\frac{27}{16}$	10. $\frac{3}{4}$
11. $\frac{1}{2}\sin 2x + C$	12. $\frac{1}{2}\sqrt{4ax + bx^4} + C$
13. $\frac{1}{3}\sin^{-1} t^3 + C$	14. $\frac{4}{3}x^3 \ln x - \frac{4}{9}x^3 + C$
15. $\frac{4}{3}\sec^3 x - 4\sec x + C$	16. $\ln\left(\sqrt{2} + 1\right)$
17. $\sqrt{\pi}$	18. $\frac{5}{6}\sqrt{6}$
19. 11	20. $\frac{\pi}{2}(1 - e^{-4})$
21A. 39,200	21B. 140,000

1.

$$f(x)=\frac{x}{x-1}\to x-1\neq 0\to x\neq 1$$

$$g\circ f(x)=\sqrt{2+\frac{x}{x-1}}=\sqrt{\frac{3x-2}{x-1}}\to\frac{3x-2}{x-1}\geq 0\to(3x-2)(x-1)\geq 0\to x\geq 1\cup x\leq\frac{2}{3}$$

$$\therefore\ x>1\cup x\leq\frac{2}{3}$$

2.

$$\because\ \lim_{x\to-2^-}\frac{x^2-4}{|x+2|}=\lim_{x\to-2^-}\frac{x^2-4}{-(x+2)}=\lim_{x\to-2^-}-(x-2)=4$$

$$\lim_{x\to-2^+}\frac{x^2-4}{|x+2|}=\lim_{x\to-2^+}\frac{x^2-4}{(x+2)}=\lim_{x\to-2^+}(x-2)=-4\qquad\therefore \text{不存在}$$

3.

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

$k=0,\ f(0)=\ln(1+0)=\ln 1=0$

$k=1,\ f'(x)=\frac{1}{1+x},\ f'(0)=\frac{1}{1+0}=1$

$k=2,\ f''(x)=-(1+x)^{-2},\ f''(0)=-1$

$k=3,\ f'''(x)=2(1+x)^{-3},\ f'''(0)=2$

$k=4,\ f^{(4)}(x)=-6(1+x)^{-4},\ f^{(4)}(0)=-6$

$$f(x)\approx f(0)+\frac{f'(0)}{1}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$
$$=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$$

4.

$$(x^2-1)=(x+1)\log 3\to x^2-(\log 3)x-(1+\log 3)=0\to x=-1,\quad 1.477$$

5.

According to the mean-value theorem

$$f'(c)=\frac{f(b)-f(a)}{b-a}\qquad \text{where}\quad c\in(a,b)$$

or

$$f'(c)=\frac{f(x+\Delta x)-f(x)}{\Delta x}\qquad \text{where}\quad c\in(x,x+\Delta x)$$

When $\Delta x\longrightarrow 0$

$$f'(x)\cong\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$f(x+\Delta x)\cong f(x)+f'(x)\Delta x$$

Let $f(x)=\sqrt[3]{x}$

then

$$f(c+\Delta x) \cong f(c) + f'(c)\Delta x$$

Let $c = 216$, $\Delta x = 0.3$

$\because$ $f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$$f(216+0.3) = \sqrt[3]{216.3} = (216)^{\frac{1}{3}} + \frac{1}{3}(216)^{-\frac{2}{3}}(0.3$$

$$= 6 + \frac{1}{3} \cdot \frac{1}{36}(0.3)$$

$$= 6 + 0.002777\ldots$$

$$\cong 6 + 0.00278 \cong 6.0028$$

$\therefore$ $\sqrt[3]{216.3} \cong \boxed{6.0028}$

6.

$h(x) = f(x^2 + g(x))$,

$f(3) = 6$, $f'(3) = -4$

$g(-2) = -1$, $g'(-2) = 2$

$\boxed{\textbf{Let } k(x) = x^2 + g(x)}$

$\Rightarrow$ to compute $h'(-2)$

$$h'(x) = \frac{dh(x)}{dx} = \frac{d(f(k(x)))}{dk(x)} \cdot \frac{dk(x)}{dx}$$

$$\because \frac{dk(x)}{dx} = \frac{d(x^2 + g(x))}{dx} = 2x + g'(x)$$

$$\text{and } \frac{d(f(k(x)))}{dk(x)} = f'(k(x))$$

$$k(-2) = (-2)^2 + g(-2) = 4 + (-1) = \mathbf{3}$$

$$f'(k(-2)) = f'(3) = -4$$

$$\Rightarrow h'(-2) = f'(k(-2)) \cdot [2(-2) + g'(-2)]$$

$$= (-4)(-4+2) = \boxed{8}$$

7.

Let

$$y = \ln f(x) = \ln x^{\pi^x} = \pi^x \ln x$$

$$\frac{dy}{dx} = \frac{\left(\frac{df(x)}{dx}\right)}{f(x)} = \left[\frac{d(\pi^x)}{dx}\right]\ln x + (\pi^x)\frac{1}{x}$$

$$\Rightarrow \frac{df(x)}{dx} = f(x) \cdot \left[(\ln\pi)(\ln x)\pi^x + \frac{\pi^x}{x}\right]$$

however

$$f(x) = x^{\pi^x}$$

$$\Rightarrow f'(x) = \frac{df(x)}{dx} = \pi^x \cdot x^{\pi^x} \cdot \left[\frac{1}{x} + (\ln\pi)(\ln x)\right]$$

8.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\cos x \cos y)\left(\tfrac{1}{2}t^{-1/2}\right) + (-\sin x \sin y)\left(-t^{-2}\right) = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y$$

9.

$$f(x) = x^4 + 2x^3 - 2x - 1$$

$$f'(x) = 4x^3 + 6x^2 - 2$$
$$= 2(2x^3 + 3x^2 - 1)$$

Since $f'(-1) = 0$ (trial & error)

It implies that $(x+1)$ is a multiplying factor of $f'(x)$.

By long division, we can factorize the derivative formula $f'(x)$.

$\therefore f'(x) = 2(2x-1)(x+1)^2$

Let $f'(c) = 0 \Rightarrow c = \frac{1}{2}$ or -1

$f''(x) = 2(6x^2 + 6x) = 12x(x+1)$

$f''(\frac{1}{2}) = 12(\frac{1}{2})(\frac{1}{2}+1) > 0 \Rightarrow f(\frac{1}{2})$: relative minimum:

$\boxed{f(\frac{1}{2}) = \frac{-27}{16}}$ (relative minimum)

Since $f''(-1) = 0 \Rightarrow$ the 2nd derivative test fails.

Return to 1st derivative test:
taking $x = -2$ and $x = 0$ **for test,**

x	-2	0
$f'(x)$	<0	<0

$\Rightarrow f(-1)$ is **not a relative extreme value.**

10.

$$f(x,y,z) = x/(y+z) = x(y+z)^{-1} \quad \Rightarrow$$

$$\nabla f(x,y,z) = \langle 1/(y+z), -x(y+z)^{-2}(1), -x(y+z)^{-2}(1) \rangle = \left\langle \frac{1}{y+z}, -\frac{x}{(y+z)^2}, -\frac{x}{(y+z)^2} \right\rangle,$$

$\nabla f(8,1,3) = \langle \frac{1}{4}, -\frac{8}{4^2}, -\frac{8}{4^2} \rangle = \langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2} \rangle$. Thus, the maximum rate of change is

$$|\nabla f(8,1,3)| = \sqrt{\tfrac{1}{16} + \tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{9}{16}} = \tfrac{3}{4}$$

11.

Let $u = 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}du$, so $\int \cos(2x)\,dx = \int \cos(u)\left(\frac{1}{2}du\right) = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(2x) + C$.

12.

Let $u = 4ax + bx^4$. Then $du = (4a + 4bx^3)\,dx = 4(a + bx^3)\,dx$, so $\int \frac{a + bx^3}{\sqrt{4ax + bx^4}}\,dx =$
$\int \frac{\frac{1}{4}du}{u^{1/2}} = \frac{1}{4}\int u^{-1/2}\,du = \frac{1}{4} \cdot 2u^{1/2} + C$
$= \frac{1}{2}\sqrt{4ax + bx^4} + C.$

13.

Let $u = t^3$. Then $du = 3t^2dt$ and $\int \frac{t^2}{\sqrt{1 - t^6}}dt = \int \frac{\frac{1}{3}du}{\sqrt{1 - u^2}}$
$= \frac{1}{3}\sin^{-1}u + C = \frac{1}{3}\sin^{-1}(t^3) + C.$

14.

Let $u = \ln(x), dv = 4x^2\,dx \;\Rightarrow\; du = dx/x, v = \frac{4}{3}x^3$. Then by equation $\int u\,dv = uv - \int v\,du$,

$$\begin{aligned}\int 4x^2 \ln(x)dx &= \tfrac{4}{3}x^3\ln(x) - \int \tfrac{4}{3}x^3(dx/x) = \tfrac{4}{3}x^3\ln(x) - \tfrac{4}{3}\int x^2\,dx \\ &= \tfrac{4}{3}x^3\ln(x) - \tfrac{4}{3}\cdot\tfrac{1}{3}x^3 + C = \tfrac{4}{3}x^3\ln(x) - \tfrac{4}{9}x^3 + C\end{aligned}$$

15.

$$\begin{aligned}\int 4\tan^3 x \sec x\,dx &= 4\int \tan^2 x \sec x \tan x\,dx = 4\int (\sec^2 x - 1)\sec x \tan x\,dx \\ &= 4\int (u^2 - 1)\,du \quad [u = \sec x,\; du = \sec x \tan x\,dx\,] \\ &= \tfrac{4}{3}u^3 - 4u + C = \tfrac{4}{3}\sec^3 x - 4\sec x + C\end{aligned}$$

16.

Let $t = 7\tan(\theta)$, so $dt = 7\sec^2(\theta)\,d\theta$, $t = 0 \Rightarrow \theta = 0$, and $t = 7 \Rightarrow \theta = \frac{\pi}{4}$. Thus,

$$\int_0^7 \frac{dt}{\sqrt{49+t^2}} = \int_0^{\pi/4} \frac{7\sec^2(\theta)\,d\theta}{\sqrt{49+49\tan^2(\theta)}} = \int_0^{\pi/4} \frac{7\sec^2(\theta)\,d\theta}{7\sec(\theta)} = \int_0^{\pi/4} \sec(\theta)\,d\theta = \Big[\ln(|\sec(\theta)+\tan(\theta)|)\Big]_0^{\pi/4}$$

$$= \ln(|\sqrt{2}+1|) - \ln(|1+0|) = \ln(\sqrt{2}+1).$$

17.

令 $A = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$\begin{aligned} A^2 &= \left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2} dy\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

改用極座標

$$\begin{aligned} A^2 &= \int_0^{2\pi}\int_0^{\infty} r\,e^{-r^2} dr d\theta \\ &= \int_0^{2\pi}\left(-\frac{1}{2}\right)\left[e^{-r^2}\right]_0^{\infty} d\theta \\ &= \int_0^{2\pi}\left(-\frac{1}{2}\right)[0-1]\,d\theta \\ &= \frac{1}{2}\times 2\pi = \pi \end{aligned}$$

所以，$A = \sqrt{\pi}$

18.

將線段寫成參數式 $x(t) = t,\ y(t) = 2t,\ z(t) = t,\ 0 \le t \le 1$

$$x^2 - y + 3z = t^2 - 2t + 3t = t^2 + t$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = \sqrt{6}dt$$

$$\begin{aligned} \int_C (x^2 - y + 3z) ds &= \int_t (t^2+t)\sqrt{6}dt \\ &= \sqrt{6}\left[\frac{1}{3}t^3 + \frac{1}{2}t^2\right]_0^1 \\ &= \sqrt{6}\left[\frac{1}{3}+\frac{1}{2}\right] \\ &= \frac{5}{6}\sqrt{6} \end{aligned}$$

19.

$$\begin{aligned} divF(x,y,z) &= \frac{\partial}{\partial x}[x^2 z] + \frac{\partial}{\partial y}[-2xz] + \frac{\partial}{\partial z}[yz] \\ &= 2xz + 0 + y \end{aligned}$$

$$divF(2,-1,3) = 2\times 2\times(3) + (-1) = 11$$

20.

$$\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^2 re^{-r^2}\,dr$$

$$= [\theta]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

21A.

$$560000 \times 5\% = 560000 \times 12\% - A \to A = 39200$$

21B.

$$1260000 \times 12\% - A = 1260000 \times 20\% - B \to B = 100800 + A = 140000$$